6.4 Vectors and Dot Product

In section 6.3, we looked at scalar multiplication of a vector.

$$a\boldsymbol{v} = a\langle v_1, v_2\rangle = \langle av_1, av_2\rangle$$

You start with a vector and end with a vector.

With the dot product, you begin with TWO vectors and end with a SCALAR.

Definition of Dot Product

$$If\ \boldsymbol{u} = \langle u_1, u_2\rangle\ and\ \boldsymbol{v} = \langle v_1, v_2\rangle,$$
$$then\ \ \boldsymbol{u}\cdot\boldsymbol{v} = u_1v_1 + u_2v_2$$

This result is a scalar (i.e. constant).

Example
Find each dot product

a) $\langle 3,4\rangle \cdot \langle 2,-3\rangle$

$3\cdot 2 + 4(-3)$

$6 + -12$

-6

b) $\langle 2,2\rangle \cdot \langle 1,-1\rangle$

$2\cdot 1 + 2(-1)$

$2 + -2$

0

c) $\langle 0,4\rangle \cdot \langle 3,-2\rangle$

$0\cdot 3 + 4(-2)$

-8

Properties of Dot Product

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $\mathbf{0} \cdot \mathbf{v} = 0$
3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
4. $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$
5. $c(\mathbf{u} \cdot \mathbf{v}) = c\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot c\mathbf{v}$

Example

$$Let\ \boldsymbol{u} = \langle 3,4 \rangle, \boldsymbol{v} = \langle -2,6 \rangle, and\ \boldsymbol{w} = \langle 1,-1 \rangle$$

$$Find\ (\boldsymbol{u} \cdot \boldsymbol{v})\boldsymbol{w}$$

$$Find\ \ \boldsymbol{u} \cdot 2\boldsymbol{v}$$

$u \cdot v = 18 \quad (u \cdot v)w = 18w = \boxed{\langle 18, -18 \rangle}$

$2v = \langle -4, 12 \rangle \quad u \cdot 2v = 3(-4) + 4 \cdot 12 = \boxed{36}$

Example

The dot product of **u** with itself is 7. What is the magnitude of **u**?

$\|u\| = \sqrt{7}$

since

$u \cdot u = \|u\|^2 = 7$

Finding the angle between two vectors

If θ is the angle between two nonzero vectors **u** and **v**, then

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}. \rightarrow \theta = \cos^{-1}\left(\frac{u\cdot v}{\|u\|\|v\|}\right)$$

Example

Find the angle between the following pairs of vectors.

$\boldsymbol{u} = \langle 3,0\rangle$ *and* $\boldsymbol{v} = \langle 1,6\rangle$

$$\cos\theta = \frac{3+0}{3\cdot\sqrt{37}} = \frac{1}{\sqrt{37}}$$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{37}}\right) \approx 80.5^\circ$$

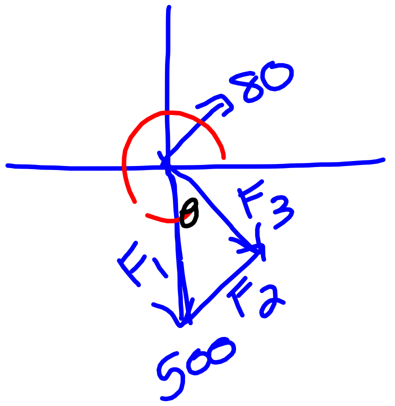
$\boldsymbol{u} = \langle 4,3\rangle$ *and* $\boldsymbol{v} = \langle 3,5\rangle$

$$\cos\theta = \frac{27}{5\cdot\sqrt{34}}$$

$$\theta \approx 21.2^\circ$$

Application Example

A Boeing 737 aircraft maintains a constant airspeed of 500 miles per hour in the direction due south. The velocity of the jet stream (air velocity) is 80 mph in a northeasterly direction. Find the actual speed and direction angle of the aircraft relative to the ground.



$$F_1 = \langle 0, -500\rangle$$

$$F_2 = \langle 40\sqrt{2}, 40\sqrt{2}\rangle$$

$$F_3 = \langle 40\sqrt{2}, -500+40\sqrt{2}\rangle$$

$$\|F_3\| \approx 447\text{mph}$$

$$\tan\theta = \frac{-500+40\sqrt{2}}{40\sqrt{2}}$$

$$\theta = 277.27^\circ$$

$$\cos\theta = \frac{F_1\cdot F_3}{\|F_1\|\cdot\|F_3\|} = \frac{221715.7288}{223500}$$

$$\theta \approx 7.24^\circ$$

The dot product can also be written as $\boldsymbol{u} \cdot \boldsymbol{v} = ||\boldsymbol{u}||||\boldsymbol{v}||cos\theta$

We can use this idea to determine if two vectors are ***orthogonal*** (perpendicular). Find the dot product. If the result is zero, then what can we infer about the angle in between the vectors?

Consider vectors $\boldsymbol{u} = \langle -12{,}30\rangle \; and \; \boldsymbol{v} = \langle \frac{5}{4}, \frac{1}{2}\rangle$

$u \cdot v = -15 + 15 = 0$

$0 = ||u|| \, ||v|| \cos\theta$

$\cos\theta = 0$

yes u & v are perpendicular

Now you examine $\boldsymbol{u} = \langle 2, -3\rangle \; and \; \boldsymbol{v} = \langle 6{,}4\rangle$

$u \cdot v = 0$

u & v are $\perp$

A word about ***parallel vectors*****.....**

Two vectors **v** and **w** are said to be parallel if there is a nonzero scalar α so that **v** = α**w**. In this case, the angle between the vectors is 0 or π.

$u \cdot V = ||u|| \, ||v|| \underbrace{\cos\theta}_{1}$

Example

Determine whether the following vectors are parallel.
v = 3**i** - **j** and **w** = 6**i**-2**j**

$w = 2v$