CHAPTER 10
Topics in Analytic Geometry

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Section 10.1 Lines

- The **inclination** of a nonhorizontal line is the positive angle θ, ($\theta < 180°$) measured counterclockwise from the x-axis to the line. A horizontal line has an inclination of zero.
- If a nonvertical line has inclination of θ and slope m, then $m = \tan\theta$.
- If two nonperpendicular lines have slopes m_1 and m_2, then the angle between the lines is given by

  $$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|.$$
- The distance between a point (x_1, y_1) and a line $Ax + By + C = 0$ is given by

  $$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}.$$

Vocabulary Check

1. inclination
2. $\tan\theta$
3. $\left|\dfrac{m_2 - m_1}{1 + m_1 m_2}\right|$
4. $\dfrac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$

1. $m = \tan\dfrac{\pi}{6} = \dfrac{\sqrt{3}}{3}$

2. $m = \tan\dfrac{\pi}{4} = 1$

3. $m = \tan\dfrac{3\pi}{4} = -1$

4. $m = \tan\dfrac{2\pi}{3} = -\sqrt{3}$

5. $m = \tan\dfrac{\pi}{3} = \sqrt{3}$

6. $m = \tan\dfrac{5\pi}{6} = -\dfrac{\sqrt{3}}{3}$

7. $m = \tan 1.27 \approx 3.2236$

8. $m = \tan 2.88 \approx -0.2677$

9. $m = -1$

$$-1 = \tan\theta$$
$$\theta = 180° + \arctan(-1)$$
$$= \frac{3\pi}{4} \text{ radians} = 135°$$

10. $-2 = \tan\theta$

$$\theta = \tan^{-1}(-2) + \pi$$
$$\approx 2.034 \text{ radians} \approx 116.6°$$

11. $m = 1$

$$1 = \tan\theta$$
$$\theta = \frac{\pi}{4} \text{ radian} = 45°$$

12. $2 = \tan\theta$

$$\theta = \tan^{-1} 2$$
$$\approx 1.107 \text{ radians} \approx 63.4°$$

13. $m = \frac{3}{4}$

$$\tfrac{3}{4} = \tan\theta$$
$$\theta = \arctan\left(\tfrac{3}{4}\right) \approx 0.6435 \text{ radian} \approx 36.9°$$

14. $-\frac{5}{2} = \tan\theta$

$$\theta = \tan^{-1}\left(-\tfrac{5}{2}\right) + \pi \approx 1.9513 \text{ radians} \approx 111.8°$$

15. (6, 1), (10, 8)

$$m = \frac{8 - 1}{10 - 6} = \frac{7}{4}$$

$$\frac{7}{4} = \tan\theta$$

$$\theta = \arctan\left(\frac{7}{4}\right) \approx 1.0517 \text{ radians} \approx 60.3°$$

16. $m = \dfrac{8 - (-3)}{12 - (-4)} = \dfrac{11}{16}$

$$\frac{11}{16} = \tan\theta$$

$$\theta = \tan^{-1}\frac{11}{16} \approx 0.6023 \text{ radian} \approx 34.5°$$

17. (−2, 20), (10, 0)

$$m = \frac{0 - 20}{10 - (-2)} = -\frac{20}{12} = -\frac{5}{3}$$

$$-\frac{5}{3} = \tan\theta$$

$$\theta = \pi + \arctan\left(-\frac{5}{3}\right) \approx 2.1112 \text{ radians} \approx 121.0°$$

18. $m = \dfrac{100 - 0}{0 - 50} = -2$

$$-2 = \tan\theta$$

$$\theta = \tan^{-1}(-2) + \pi \approx 2.0344 \text{ radians} \approx 116.6°$$

19. $6x - 2y + 8 = 0$

$$y = 3x + 4 \implies m = 3$$

$$3 = \tan\theta$$

$$\theta = \arctan 3 \approx 1.2490 \text{ radians} \approx 71.6°$$

20. $4x + 5y - 9 = 0$

$$y = -\tfrac{4}{5}x + \tfrac{9}{5} \implies m = -\tfrac{4}{5}$$

$$-\tfrac{4}{5} = \tan\theta$$

$$\theta = \tan^{-1}\left(-\tfrac{4}{5}\right) + \pi$$

$$\approx 2.4669 \text{ radians} \approx 141.3°$$

21. $5x + 3y = 0$

$$y = -\frac{5}{3}x \implies m = -\frac{5}{3}$$

$$-\frac{5}{3} = \tan\theta$$

$$\theta = \pi + \arctan\left(-\frac{5}{3}\right) \approx 2.1112 \text{ radians} \approx 121.0°$$

22. $x - y - 10 = 0$

$$y = x - 10 \implies m = 1$$

$$1 = \tan\theta$$

$$\theta = \tan^{-1} 1 = 45° = \frac{\pi}{4} \text{ radian}$$

23. $3x + y = 3 \implies y = -3x + 3 \implies m_1 = -3$

$$x - y = 2 \implies y = x - 2 \implies m_2 = 1$$

$$\tan\theta = \left|\frac{1 - (-3)}{1 + (-3)(1)}\right| = 2$$

$$\theta = \arctan 2 \approx 1.1071 \text{ radians} \approx 63.4°$$

24. $x + 3y = 2 \implies y = -\dfrac{1}{3}x + \dfrac{2}{3} \implies m_1 = -\dfrac{1}{3}$

$$x - 2y = -3 \implies y = \frac{1}{2}x + \frac{3}{2} \implies m_2 = \frac{1}{2}$$

$$\tan\theta = \left|\frac{(1/2) - (-1/3)}{1 + (-1/3)(1/2)}\right| = 1$$

$$\theta = \tan^{-1} 1 = 45° = \frac{\pi}{4} \text{ radian}$$

25. $x - y = 0 \implies y = x \implies m_1 = 1$

$$3x - 2y = -1 \implies y = \frac{3}{2}x + \frac{1}{2} \implies m_2 = \frac{3}{2}$$

$$\tan\theta = \left|\frac{\frac{3}{2} - 1}{1 + \left(\frac{3}{2}\right)(1)}\right| = \frac{1}{5}$$

$$\theta = \arctan\frac{1}{5} \approx 0.1974 \text{ radian} \approx 11.3°$$

26. $2x - y = 2 \implies y = 2x - 2 \implies m_1 = 2$

$$4x + 3y = 24 \implies y = -\frac{4}{3}x + 8 \implies m_2 = -\frac{4}{3}$$

$$\tan\theta = \left|\frac{(-4/3) - 2}{1 + (2)(-4/3)}\right| = 2$$

$$\theta = \tan^{-1} 2 \approx 63.4° \approx 1.1071 \text{ radians}$$

27. $x - 2y = 7 \Rightarrow y = \frac{1}{2}x - \frac{7}{2} \Rightarrow m_1 = \frac{1}{2}$

$6x + 2y = 5 \Rightarrow y = -3x + \frac{5}{2} \Rightarrow m_2 = -3$

$$\tan\theta = \left|\frac{-3 - \frac{1}{2}}{1 + \left(\frac{1}{2}\right)(-3)}\right| = 7$$

$\theta = \arctan 7 \approx 1.4289 \text{ radians} \approx 81.9°$

28. $5x + 2y = 16 \Rightarrow y = -\frac{5}{2}x + 8 \Rightarrow m_1 = -\frac{5}{2}$

$3x - 5y = -1 \Rightarrow y = \frac{3}{5}x + \frac{1}{5} \Rightarrow m_2 = \frac{3}{5}$

$$\tan\theta = \left|\frac{(-5/2) - (3/5)}{1 + (-5/2)(3/5)}\right| = \frac{31}{5}$$

$\theta = \tan^{-1}\frac{31}{5} \approx 80.8° \approx 1.4109 \text{ radians}$

29. $x + 2y = 8 \Rightarrow y = -\frac{1}{2}x + 4 \Rightarrow m_1 = -\frac{1}{2}$

$x - 2y = 2 \Rightarrow y = \frac{1}{2}x - 1 \Rightarrow m_2 = \frac{1}{2}$

$$\tan\theta = \left|\frac{\frac{1}{2} - \left(-\frac{1}{2}\right)}{1 + \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)}\right| = \frac{4}{3}$$

$\theta = \arctan\left(\frac{4}{3}\right) \approx 0.9273 \text{ radian} \approx 53.1°$

30. $3x - 5y = 3 \Rightarrow y = \frac{3}{5}x - \frac{3}{5} \Rightarrow m_1 = \frac{3}{5}$

$3x + 5y = 12 \Rightarrow y = -\frac{3}{5}x + \frac{12}{5} \Rightarrow m_2 = -\frac{3}{5}$

$$\tan\theta = \left|\frac{(3/5) - (-3/5)}{1 + (3/5)(-3/5)}\right| = \frac{15}{8}$$

$\theta = \tan^{-1}\frac{15}{8} \approx 61.9° \approx 1.0808 \text{ radians}$

31. $0.05x - 0.03y = 0.21 \Rightarrow y = \frac{5}{3}x - 7 \Rightarrow m_1 = \frac{5}{3}$

$0.07x + 0.02y = 0.16 \Rightarrow y = -\frac{7}{2}x + 8 \Rightarrow m_2 = -\frac{7}{2}$

$$\tan\theta = \left|\frac{\left(-\frac{7}{2}\right) - \left(\frac{5}{3}\right)}{1 + \left(\frac{5}{3}\right)\left(-\frac{7}{2}\right)}\right| = \frac{31}{29}$$

$\theta = \arctan\left(\frac{31}{29}\right) \approx 0.8187 \text{ radian} \approx 46.9°$

32. $0.02x - 0.05y = -0.19 \Rightarrow y = \frac{2}{5}x + \frac{19}{5} \Rightarrow m_1 = \frac{2}{5}$

$0.03x + 0.04y = 0.52 \Rightarrow y = -\frac{3}{4}x + 13 \Rightarrow m_2 = -\frac{3}{4}$

$$\tan\theta = \left|\frac{(-3/4) - (2/5)}{1 + (2/5)(-3/4)}\right| \approx \frac{23}{14}$$

$\theta = \tan^{-1}\left(\frac{23}{14}\right) \approx 58.7° \approx 1.0240 \text{ radians}$

33. Let $A = (2, 1)$, $B = (4, 4)$, and $C = (6, 2)$.

Slope of AB: $m_1 = \frac{1 - 4}{2 - 4} = \frac{3}{2}$

Slope of BC: $m_2 = \frac{4 - 2}{4 - 6} = -1$

Slope of AC: $m_3 = \frac{1 - 2}{2 - 6} = \frac{1}{4}$

$$\tan A = \left|\frac{\frac{1}{4} - \frac{3}{2}}{1 + \left(\frac{3}{2}\right)\left(\frac{1}{4}\right)}\right| = \frac{\frac{5}{4}}{\frac{11}{8}} = \frac{10}{11}$$

$A = \arctan\left(\frac{10}{11}\right) \approx 42.3°$

$$\tan B = \left|\frac{\frac{3}{2} - (-1)}{1 + (-1)\left(\frac{3}{2}\right)}\right| = \frac{\frac{5}{2}}{\frac{1}{2}} = 5$$

$B = \arctan 5 \approx 78.7°$

$$\tan C = \left|\frac{-1 - \frac{1}{4}}{1 + \left(\frac{1}{4}\right)(-1)}\right| = \frac{\frac{5}{4}}{\frac{3}{4}} = \frac{5}{3}$$

$C = \arctan\left(\frac{5}{3}\right) \approx 59.0°$

34. Let $A = (-3, 2)$, $B = (1, 3)$, and $C = (2, 0)$.

Slope of AB: $m_1 = \frac{2 - 3}{-3 - 1} = \frac{1}{4}$

Slope of BC: $m_2 = \frac{3 - 0}{1 - 2} = -3$

Slope of AC: $m_3 = \frac{2 - 0}{-3 - 2} = -\frac{2}{5}$

$$\tan A = \left|\frac{(1/4) - (-2/5)}{1 + (-2/5)(1/4)}\right| = \frac{13/20}{18/20} = \frac{13}{18}$$

$A = \tan^{-1}\left(\frac{13}{18}\right) \approx 35.8°$

$$\tan C = \left|\frac{-3 - (-2/5)}{1 + (-3)(-2/5)}\right| \approx \frac{13/5}{11/5} = \frac{13}{11}$$

$C = \tan^{-1}\left(\frac{13}{11}\right) \approx 49.8°$

$$B = 180° - A - C \approx 180° - 35.8° - 49.8° = 94.4°$$

35. Let $A = (-4, -1)$, $B = (3, 2)$, and $C = (1, 0)$.

Slope of AB: $m_1 = \dfrac{-1-2}{-4-3} = \dfrac{3}{7}$

Slope of BC: $m_2 = \dfrac{2-0}{3-1} = 1$

Slope of AC: $m_3 = \dfrac{-1-0}{-4-1} = \dfrac{1}{5}$

$$\tan A = \left|\frac{\frac{1}{5}-\frac{3}{7}}{1+\left(\frac{3}{7}\right)\left(\frac{1}{5}\right)}\right| = \frac{\frac{8}{35}}{\frac{38}{35}} = \frac{4}{19}$$

$$A = \arctan\left(\frac{4}{19}\right) \approx 11.9°$$

$$\tan B = \left|\frac{1-\frac{3}{7}}{1+\left(\frac{3}{7}\right)(1)}\right| = \frac{\frac{4}{7}}{\frac{10}{7}} = \frac{2}{5}$$

$$B = \arctan\left(\frac{2}{5}\right) \approx 21.8°$$

$$C = 180° - A - B$$
$$\approx 180° - 11.9° - 21.8° = 146.3°$$

36. Let $A = (-3, 4)$, $B = (2, 1)$, and $C = (-2, 2)$.

Slope of AB: $m_1 = \dfrac{4-1}{-3-2} = -\dfrac{3}{5}$

Slope of BC: $m_2 = \dfrac{1-2}{2-(-2)} = -\dfrac{1}{4}$

Slope of AC: $m_3 = \dfrac{4-2}{-3-(-2)} = -2$

$$\tan A = \left|\frac{(-3/5)-(-2)}{1+(-3/5)(-2)}\right| = \frac{7}{11}$$

$$A = \tan^{-1}\left(\frac{7}{11}\right) \approx 32.5°$$

$$\tan B = \left|\frac{(-3/5)-(-1/4)}{1+(-3/5)(-1/4)}\right| = \frac{7}{23}$$

$$B = \tan^{-1}\left(\frac{7}{23}\right) \approx 16.9°$$

$$C = 180° - A - B \approx 180° - 32.5° - 16.9°$$
$$= 130.6°$$

37. $(0, 0) \implies x_1 = 0$ and $y_1 = 0$

$4x + 3y = 0 \implies A = 4, B = 3$, and $C = 0$

$$d = \frac{|4(0) + 3(0) + 0|}{\sqrt{4^2 + 3^2}} = \frac{0}{5} = 0$$

Note: The point is *on* the line.

38. $(0, 0) \implies x_1 = 0$ and $y_1 = 0$

$2x - y - 4 = 0 \implies A = 2, B = -1$, and $C = -4$

$$d = \frac{|2(0) + (-1)(0) + (-4)|}{\sqrt{2^2 + (-1)^2}}$$
$$= \frac{4}{\sqrt{5}} = \frac{4\sqrt{5}}{5} \approx 1.7889$$

39. $(2, 3) \implies x_1 = 2$ and $y_1 = 3$

$4x + 3y - 10 = 0 \implies A = 4, B = 3$, and $C = -10$

$$d = \frac{|4(2) + 3(3) + (-10)|}{\sqrt{4^2 + 3^2}} = \frac{7}{5}$$

40. $(-2, 1) \implies x_1 = -2$ and $y_1 = 1$

$x - y - 2 = 0 \implies A = 1, B = -1$, and $C = -2$

$$d = \frac{|1(-2) + (-1)(1) + (-2)|}{\sqrt{1^2 + (-1)^2}}$$
$$= \frac{5}{\sqrt{2}} = \frac{5\sqrt{2}}{2} \approx 3.5355$$

41. $(6, 2) \implies x_1 = 6$ and $y_1 = 2$

$x + 1 = 0 \implies A = 1, B = 0$, and $C = 1$

$$d = \frac{|1(6) + 0(2) + 1|}{\sqrt{1^2 + 0^2}} = 7$$

42. $(10, 8) \implies x_1 = 10$ and $y_1 = 8$

$y - 4 = 0 \implies A = 0, B = 1$, and $C = -4$

$$d = \frac{|0(10) + 1(8) + (-4)|}{\sqrt{0^2 + 1^2}} = \frac{4}{1} = 4$$

43. $(0, 8) \implies x_1 = 0$ and $y_1 = 8$

$6x - y = 0 \implies A = 6, B = -1$, and $C = 0$

$$d = \frac{|6(0) + (-1)(8) + 0|}{\sqrt{6^2 + (-1)^2}}$$
$$= \frac{8}{\sqrt{37}} = \frac{8\sqrt{37}}{37} \approx 1.3152$$

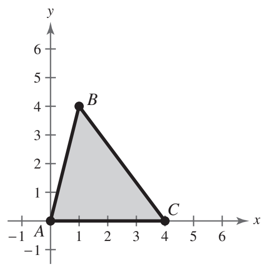
44. $(4, 2) \implies x_1 = 4$ and $y_1 = 2$

$x - y - 20 = 0 \implies A = 1, B = -1$, and $C = -20$

$$d = \frac{|1(4) + (-1)(2) + (-20)|}{\sqrt{1^2 + (-1)^2}}$$
$$= \frac{18}{\sqrt{2}} = 9\sqrt{2} \approx 12.7279$$

45. $A = (0, 0), B = (1, 4), C = (4, 0)$

(a)



(b) The slope the line through AC is $m = \dfrac{0 - 0}{4 - 0} = 0.$

The equation of the line through AC is $y = 0.$

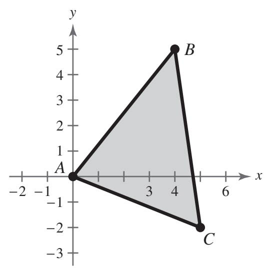
The distance between the line and $B = (1, 4)$ is

$$d = \frac{|0(1) + (1)(4) + 0|}{\sqrt{0^2 + 1^2}} = 4.$$

(c) The distance between A and C is 4.

$$A = \frac{1}{2}(4)(4) = 8 \text{ square units}$$

46. (a)



(b) The slope of the line through AC is $m = \dfrac{0 + 2}{0 - 5} = -\dfrac{2}{5}.$

The equation of the line is $y - 0 = -\dfrac{2}{5}(x - 0) \implies 2x + 5y = 0.$

The altitude from vertex B to side AC is the distance between the line through AC and

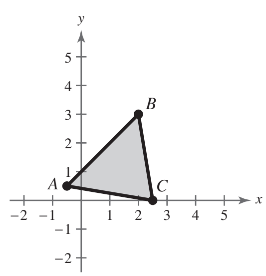
$$B = (4, 5) \implies d = \frac{|2(4) + 5(5) + 0|}{\sqrt{2^2 + 5^2}} = \frac{33}{\sqrt{29}} = \frac{33\sqrt{29}}{29}.$$

(c) The distance between A and C is $d = \sqrt{(0 - 5)^2 + (0 + 2)^2} = \sqrt{29},$ which is the length of the base of the triangle. So,

$$A = \frac{1}{2}\sqrt{29}\left(\frac{33\sqrt{29}}{29}\right) = \frac{33}{2} \text{ square units.}$$

47. $A = \left(-\dfrac{1}{2}, \dfrac{1}{2}\right), B = (2, 3), C = \left(\dfrac{5}{2}, 0\right)$

(a)



(b) The slope of the line through AC is $m = \dfrac{\frac{1}{2} - 0}{\left(-\frac{1}{2}\right) - \frac{5}{2}} = -\dfrac{1}{6}.$

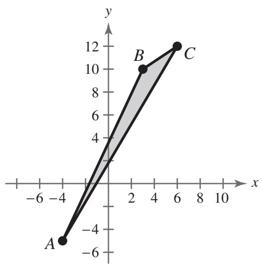
The equation of the line through AC is $y - 0 = -\dfrac{1}{6}\left(x - \dfrac{5}{2}\right) \implies 2x + 12y - 5 = 0.$

The distance between the line and $B = (2, 3)$ is $d = \dfrac{|2(2) + 12(3) + (-5)|}{\sqrt{2^2 + 12^2}} = \dfrac{35}{\sqrt{148}} = \dfrac{35\sqrt{37}}{74}.$

(c) The distance between A and C is $d = \sqrt{\left[\left(-\dfrac{1}{2}\right) - \left(\dfrac{5}{2}\right)\right]^2 + \left[\left(\dfrac{1}{2}\right) - 0\right]^2} = \dfrac{\sqrt{37}}{2}.$

$$A = \frac{1}{2}\left(\frac{\sqrt{37}}{2}\right)\left(\frac{35\sqrt{37}}{74}\right) = \frac{35}{8} \text{ square units}$$

48. (a)



(b) The slope of the line through AC is $m = \dfrac{12 - (-5)}{6 - (-4)} = \dfrac{17}{10}$.

The equation of the line through AC is $y - 12 = \dfrac{17}{10}(x - 6) \implies 17x - 10y + 18 = 0$.

The altitude from vertex B to side AC is the distance between the line through AC and

$B = (3, 10) \implies d = \dfrac{|17(3) + (-10)(10) + 18|}{\sqrt{17^2 + (-10)^2}} = \dfrac{31}{\sqrt{389}} = \dfrac{31\sqrt{389}}{389}$.

(c) The distance between A and C is $d = \sqrt{(6 + 4)^2 + (12 + 5)^2} = \sqrt{389}$, which is the length of the base of the triangle.

$$A = \frac{1}{2}\left(\sqrt{389}\right)\left(\frac{31\sqrt{389}}{389}\right) = \frac{31}{2}$$

49. $x + y = 1 \implies (0, 1)$ is a point on the line $\implies x_1 = 0$ and $y_1 = 1$

$x + y = 5 \implies A = 1, B = 1$, and $C = -5$

$$d = \frac{|1(0) + 1(1) + (-5)|}{\sqrt{1^2 + 1^2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

50. $3x - 4y = 1$

$3x - 4y = 10$

A point on $3x - 4y = 10$ is $\left(0, -\frac{5}{2}\right)$. The distance between $\left(0, -\frac{5}{2}\right)$ and $3x - 4y = 1$ is:

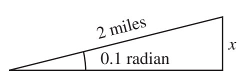
$A = 3, B = -4, C = -1, x_1 = 0, y_1 = -\dfrac{5}{2}$

$$d = \frac{|3(0) + (-4)(-5/2) - 1|}{\sqrt{3^2 + (-4)^2}} = \frac{9}{5}$$

51. Slope: $m = \tan 0.1 \approx 0.1003$

Change in elevation: $\sin 0.1 = \dfrac{x}{2(5280)}$

$x \approx 1054$ feet



Not drawn to scale

52. Slope: $m = \tan 0.2 \approx 0.2027$

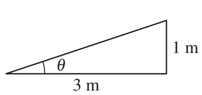
Change in elevation:

$\sin 0.20 = \dfrac{x}{5280} \implies x = 5280 \sin 0.20 \approx 1049$ feet

53. Slope $= \dfrac{3}{5}$

Inclination $= \tan^{-1}\dfrac{3}{5} \approx 31.0°$

54. (a)



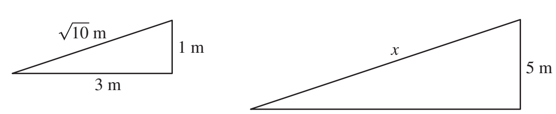
(b) $m = \dfrac{1}{3}$

$\dfrac{1}{3} = \tan\theta$

$\tan^{-1}\left(\dfrac{1}{3}\right) = \theta$

or $\theta \approx 18.4°$

(c) Use similar triangles:



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$\dfrac{x}{5} = \dfrac{\sqrt{10}}{1}$

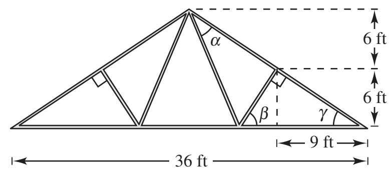
$x = 5\sqrt{10} \approx 15.8$ m

55. $\tan \gamma = \dfrac{6}{9}$

$$\gamma = \arctan\left(\frac{2}{3}\right) \approx 33.69°$$

$$\beta = 90 - \gamma \approx 56.31°$$

Also, since the right triangles containing α and β are equal, $\alpha = \gamma \approx 33.69°$.



56. (a)

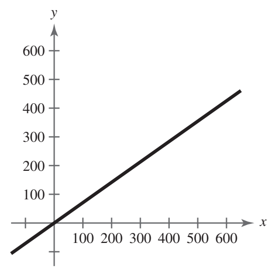
$$m = \tan$$
$$0.709 = \tan \theta$$
$$\tan^{-1} 0.709 = \theta$$
$$\theta \approx 0.6167 \text{ radian, or } 35.34°$$

(b)

$$\sin \theta = \frac{\text{elev } \Delta}{896.5}$$
$$896.5 \sin \theta = \text{elev } \Delta$$
$$\text{elev } \Delta = 896.5 \sin 0.6167 \approx 518.5 \text{ ft}$$

(c) $m = 0.709$ and y-intercept $= (0, 0)$, so $y = 0.709x$.

(d)



57. True. The inclination of a line is related to its slope by $m = \tan \theta$. If the angle is greater than $\pi/2$ but less than π, then the angle is in the second quadrant where the tangent function is negative.

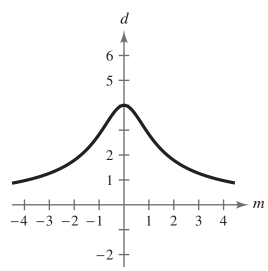
58. False. Substitute $m_1 = \tan \theta_1$ and $m_2 = \tan \theta_2$ into the formula for the angle between two lines.

59. (a) $(0, 0) \implies x_1 = 0$ and $y_1 = 0$

$$y = mx + 4 \implies 0 = mx - y + 4$$

$$d = \frac{|m(0) + (-1)(0) + 4|}{\sqrt{m^2 + (-1)^2}} = \frac{4}{\sqrt{m^2 + 1}}$$

(b)



(c) The maximum distance of 4 occurs when the slope m is 0 and the line through $(0, 4)$ is horizontal.

(d) The graph has a horizontal asymptote at $d = 0$. As the slope becomes larger, the distance between the origin and the line, $y = mx + 4$, becomes smaller and approaches 0.

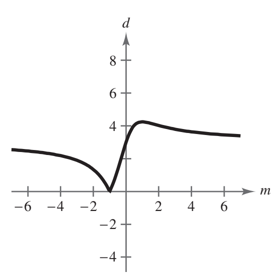
60. Slope m and y-intercept $(0, 4)$

(a) $(x_1, y_1) = (3, 1)$ and line: $y = mx + 4$

$$A = -m, B = 1, C = -4$$

$$d = \frac{|(-m)(3) + (1)(1) + (-4)|}{\sqrt{(-m)^2 + 1^2}} = \frac{3|m + 1|}{\sqrt{m^2 + 1}}$$

(b)



(c) From the graph it appears that the maximum distance is obtained when $m = 1$.

(d) From the graph it appears that the distance is 0 when $m = -1$.

(e) The asymptote of the graph in part (b) is $d = 3$. As the line approaches the vertical, the distance approaches 3.

61. $f(x) = (x - 7)^2$

x-intercept: $0 = (x - 7)^2 \implies x = 7$

$(7, 0)$

y-intercept: $y = (0 - 7)^2 = 49$

$(0, 49)$

62. $f(x) = (x + 9)^2$

$f(x) = (x + 9)^2 = 0 \implies x = -9$

x-intercept: $(-9, 0)$

$f(0) = (0 + 9)^2 = 81$

y-intercept: $(0, 81)$

63. $f(x) = (x - 5)^2 - 5$

x-intercepts: $0 = (x - 5)^2 - 5$

$5 = (x - 5)^2$

$\pm\sqrt{5} = x - 5$

$5 \pm \sqrt{5} = x$

$\left(5 \pm \sqrt{5}, 0\right)$

y-intercept: $y = (0 - 5)^2 - 5 = 20$

$(0, 20)$

64. $f(x) = (x + 11)^2 + 12$

$f(x) = (x + 11)^2 + 12 = 0$

$(x + 11)^2 = -12$

No solution

x-intercept: none

$f(0) = (0 + 11)^2 + 12 = 133$

y-intercept: $(0, 133)$

65. $f(x) = x^2 - 7x - 1$

x-intercepts: $0 = x^2 - 7x - 1$

$x = \dfrac{7 \pm \sqrt{53}}{2}$ by the Quadratic Formula

$\left(\dfrac{7 \pm \sqrt{53}}{2}, 0\right)$

y-intercept: $y = 0^2 - 7(0) - 1 = -1$

$(0, -1)$

66. $f(x) = x^2 + 9x - 22$

$f(x) = x^2 + 9x - 22 = 0$

$(x + 11)(x - 2) = 0$

$x = -11, 2$

x-intercepts: $(-11, 0), (2, 0)$

$f(0) = -22$

y-intercept: $(0, -22)$

67.
$$\begin{aligned} f(x) &= 3x^2 + 2x - 16 \\ &= 3\left(x^2 + \tfrac{2}{3}x\right) - 16 \\ &= 3\left(x^2 + \tfrac{2}{3}x + \tfrac{1}{9}\right) - \tfrac{1}{3} - 16 \\ &= 3\left(x + \tfrac{1}{3}\right)^2 - \tfrac{49}{3} \end{aligned}$$

Vertex: $\left(-\tfrac{1}{3}, -\tfrac{49}{3}\right)$

68.
$$\begin{aligned} f(x) &= 2x^2 - x - 21 \\ &= 2\left[x^2 - \tfrac{1}{2}x - \tfrac{21}{2}\right] = 2\left[x^2 - \tfrac{1}{2}x + \tfrac{1}{16} - \tfrac{1}{16} - \tfrac{21}{2}\right] \\ &= 2\left[\left(x - \tfrac{1}{4}\right)^2 - \tfrac{169}{16}\right] \\ &= 2\left(x - \tfrac{1}{4}\right)^2 - \tfrac{169}{8} \end{aligned}$$

Vertex: $\left(\tfrac{1}{4}, -\tfrac{169}{8}\right)$

69.
$$\begin{aligned} f(x) &= 5x^2 + 34x - 7 \\ &= 5\left(x^2 + \tfrac{34}{5}x\right) - 7 \\ &= 5\left(x^2 + \tfrac{34}{5}x + \tfrac{289}{25}\right) - \tfrac{289}{5} - 7 \\ &= 5\left(x + \tfrac{17}{5}\right)^2 - \tfrac{324}{5} \end{aligned}$$

Vertex: $\left(-\tfrac{17}{5}, -\tfrac{324}{5}\right)$

70.
$$\begin{aligned} f(x) &= -x^2 - 8x - 15 \\ &= -[x^2 + 8x + 15] = -[x^2 + 8x + 16 - 16 + 15] \\ &= -[(x + 4)^2 - 1] = -(x + 4)^2 + 1 \end{aligned}$$

Vertex: $(-4, 1)$

71.
$$\begin{aligned} f(x) &= 6x^2 - x - 12 \\ &= 6\left(x^2 - \tfrac{1}{6}x\right) - 12 \\ &= 6\left(x^2 - \tfrac{1}{6}x + \tfrac{1}{144}\right) - \tfrac{1}{24} - 12 \\ &= 6\left(x - \tfrac{1}{12}\right)^2 - \tfrac{289}{24} \end{aligned}$$

Vertex: $\left(\tfrac{1}{12}, -\tfrac{289}{24}\right)$

72.
$$\begin{aligned} f(x) &= -8x^2 - 34x - 21 \\ &= -8\left[x^2 + \tfrac{17}{4}x + \tfrac{21}{8}\right] \\ &= -8\left[x^2 + \tfrac{17}{4}x + \tfrac{289}{64} - \tfrac{289}{64} + \tfrac{21}{8}\right] \\ &= -8\left[\left(x + \tfrac{17}{8}\right)^2 - \tfrac{121}{64}\right] \\ &= -8\left(x + \tfrac{17}{8}\right)^2 + \tfrac{121}{8} \end{aligned}$$

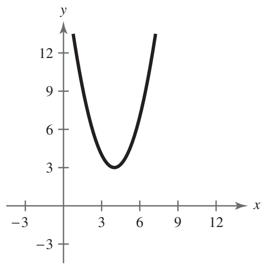
Vertex: $\left(-\tfrac{17}{8}, \tfrac{121}{8}\right)$

73. $f(x) = (x - 4)^2 + 3$

Vertex: $(4, 3)$

y-intercept: $(0, 19)$

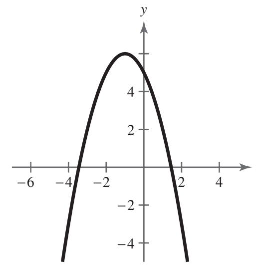
x-intercept: None



74. $f(x) = 6 - (x + 1)^2$

Vertex: $(-1, 6)$

x	-4	-3	-2	-1	0	1	2
$g(x)$	-3	2	5	6	5	2	-3



75. $g(x) = 2x^2 - 3x + 1$

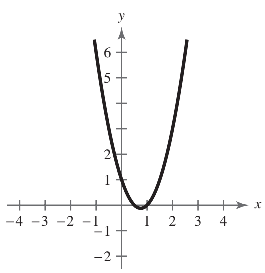
$$= 2\left(x^2 - \frac{3}{2}x + \frac{9}{16}\right) - \frac{9}{8} + 1$$

$$= 2\left(x - \frac{3}{4}\right)^2 - \frac{1}{8}$$

Vertex: $\left(\frac{3}{4}, -\frac{1}{8}\right)$

y-intercept: $(0, 1)$

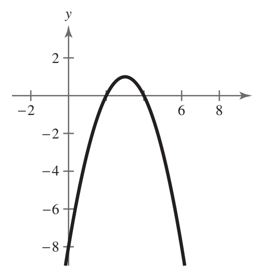
x-intercept: $\left(\frac{1}{2}, 0\right), (1, 0)$



76. $g(x) = -x^2 + 6x - 8$

$\dfrac{-b}{2a} = \dfrac{-6}{2(-1)} = 3 \implies \text{Vertex} = (3, g(3)) = (3, 1)$

x	0	1	2	3	4	5	6
$g(x)$	-8	-3	0	1	0	-3	-8



Section 10.2 Introduction to Conics: Parabolas

- A **parabola** is the set of all points (x, y) that are equidistant from a fixed line **(directrix)** and a fixed point **(focus)** not on the line.
- The standard equation of a parabola with vertex (h, k) and:
  (a) Vertical axis $x = h$ and directrix $y = k - p$ is: $(x - h)^2 = 4p(y - k), p \neq 0$
  (b) Horizontal axis $y = k$ and directrix $x = h - p$ is: $(y - k)^2 = 4p(x - h), p \neq 0$
- The tangent line to a parabola at a point P makes **equal angles** with:
  (a) the line through P and the focus.
  (b) the axis of the parabola.

Vocabulary Check

1. conic

2. locus

3. parabola; directrix; focus

4. axis

5. vertex

6. focal chord

7. tangent

1. A circle is formed when a plane intersects the top or bottom half of a double-napped cone and is perpendicular to the axis of the cone.

2. An ellipse is formed when a plane intersects only the top or bottom half of a double-napped cone but is not perpendicular to the axis of the cone, not parallel to the side of the cone, and does not intersect the vertex.

3. A parabola is formed when a plane intersects the top or bottom half of a double-napped cone, is parallel to the side of the cone, and does not intersect the vertex.

4. A hyperbola is formed when a plane intersects both halves of a double-napped cone, is parallel to the axis of the cone, and does not intersect the vertex.

5. $y^2 = -4x$
Vertex: $(0, 0)$
Opens to the left since p is negative; matches graph (e).

6. $x^2 = 2y$
Vertex: $(0, 0)$
$p = \frac{1}{2} > 0$
Opens upward; matches graph (b).

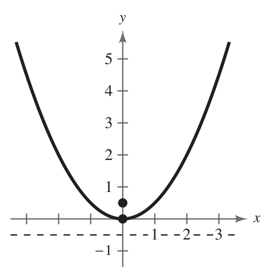
7. $x^2 = -8y$
Vertex: $(0, 0)$
Opens downward since p is negative; matches graph (d).

8. $y^2 = -12x$
Vertex: $(0, 0)$
$p = -3 < 0$
Opens to the left; matches graph (f).

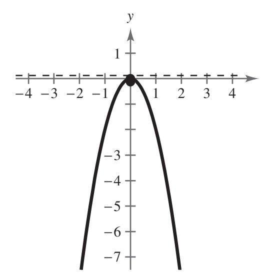
9. $(y - 1)^2 = 4(x - 3)$
Vertex: $(3, 1)$
Opens to the right since p is positive; matches graph (a).

10. $(x + 3)^2 = -2(y - 1)$
Vertex: $(-3, 1)$
$p = -\frac{1}{2} < 0$
Opens downward; matches graph (c).

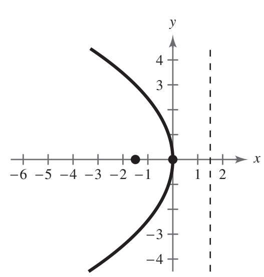
11. $y = \frac{1}{2}x^2$
$x^2 = 2y$
$x^2 = 4\left(\frac{1}{2}\right)y \implies h = 0, k = 0, p = \frac{1}{2}$
Vertex: $(0, 0)$
Focus: $\left(0, \frac{1}{2}\right)$
Directrix: $y = -\frac{1}{2}$



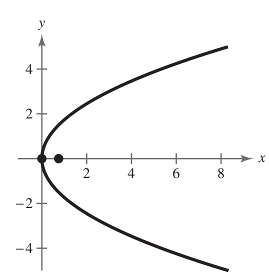
12. $y = -2x^2 \implies x^2 = 4\left(-\frac{1}{8}\right)y$
Vertex: $(0, 0)$
Focus: $\left(0, -\frac{1}{8}\right)$
Directrix: $y = \frac{1}{8}$



13. $y^2 = -6x$
$y^2 = 4\left(-\frac{3}{2}\right)x \implies h = 0, k = 0, p = -\frac{3}{2}$
Vertex: $(0, 0)$
Focus: $\left(-\frac{3}{2}, 0\right)$
Directrix: $x = \frac{3}{2}$



14. $y^2 = 3x \implies 4\left(\frac{3}{4}\right)x$
Vertex: $(0, 0)$
Focus: $\left(\frac{3}{4}, 0\right)$
Directrix: $x = -\frac{3}{4}$



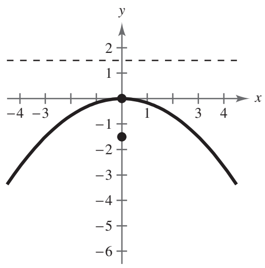
15. $x^2 + 6y = 0$

$$x^2 = -6y = 4\left(-\tfrac{3}{2}\right)y \implies h = 0, k = 0, p = -\tfrac{3}{2}$$

Vertex: $(0, 0)$

Focus: $\left(0, -\tfrac{3}{2}\right)$

Directrix: $y = \tfrac{3}{2}$



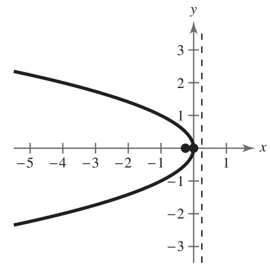
16. $x + y^2 = 0$

$$y^2 = -x = 4\left(-\tfrac{1}{4}\right)x$$

Vertex: $(0, 0)$

Focus: $\left(-\tfrac{1}{4}, 0\right)$

Directrix: $x = \tfrac{1}{4}$



17. $(x - 1)^2 + 8(y + 2) = 0$

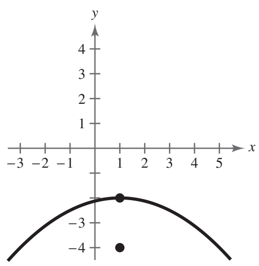
$$(x - 1)^2 = 4(-2)(y + 2)$$

$h = 1, k = -2, p = -2$

Vertex: $(1, -2)$

Focus: $(1, -4)$

Directrix: $y = 0$



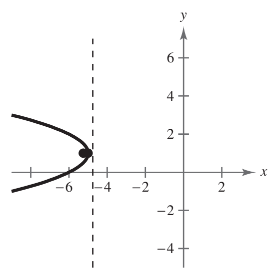
18. $(x + 5) + (y - 1)^2 = 0$

$$(y - 1)^2 = 4\left(-\tfrac{1}{4}\right)(x + 5)$$

Vertex: $(-5, 1)$

Focus: $\left(-5 + \left(-\tfrac{1}{4}\right), 1\right) \implies \left(-\tfrac{21}{4}, 1\right)$

Directrix: $x = -5 - \left(-\tfrac{1}{4}\right) = -\tfrac{19}{4}$



19. $\left(x + \tfrac{3}{2}\right)^2 = 4(y - 2)$

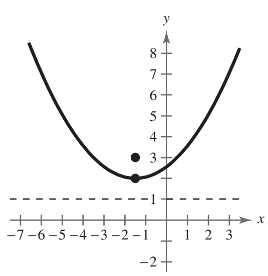
$\left(x + \tfrac{3}{2}\right)^2 = 4(1)(y - 2)$

$h = -\tfrac{3}{2}, k = 2, p = 1$

Vertex: $\left(-\tfrac{3}{2}, 2\right)$

Focus: $\left(-\tfrac{3}{2}, 3\right)$

Directrix: $y = 1$

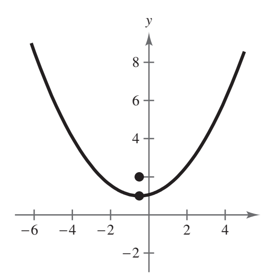


20. $\left(x + \tfrac{1}{2}\right)^2 = 4(y - 1) = 4(1)(y - 1)$

Vertex: $\left(-\tfrac{1}{2}, 1\right)$

Focus: $\left(-\tfrac{1}{2}, 1 + 1\right) \implies \left(-\tfrac{1}{2}, 2\right)$

Directrix: $y = 1 - 1 = 0$



21.

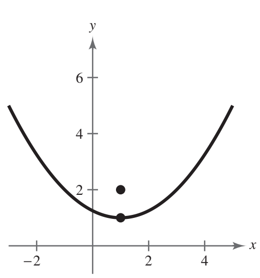
$$y = \tfrac{1}{4}(x^2 - 2x + 5)$$
$$4y = x^2 - 2x + 5$$
$$4y - 5 + 1 = x^2 - 2x + 1$$
$$4y - 4 = (x - 1)^2$$
$$(x - 1)^2 = 4(1)(y - 1)$$

$h = 1, k = 1, p = 1$

Vertex: $(1, 1)$

Focus: $(1, 2)$

Directrix: $y = 0$



22.

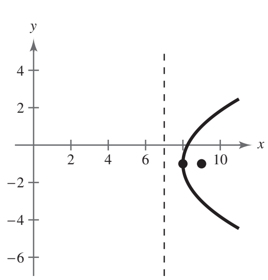
$$x = \tfrac{1}{4}(y^2 + 2y + 33)$$
$$4x = y^2 + 2y + 1 - 1 + 33 = (y + 1)^2 + 32$$

$(y + 1)^2 = 4(1)(x - 8)$

Vertex: $(8, -1)$

Focus: $(9, -1)$

Directrix: $x = 7$



23. $y^2 + 6y + 8x + 25 = 0$

$$y^2 + 6y + 9 = -8x - 25 + 9$$

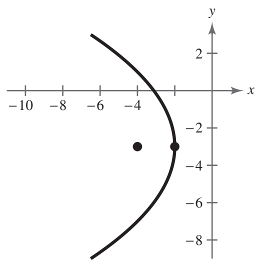
$$(y + 3)^2 = 4(-2)(x + 2)$$

$h = -2, k = -3, p = -2$

Vertex: $(-2, -3)$

Focus: $(-4, -3)$

Directrix: $x = 0$



24. $y^2 - 4y - 4x = 0$

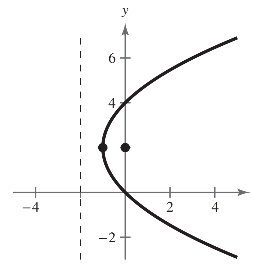
$$y^2 - 4y + 4 = 4x + 4$$

$$(y - 2)^2 = 4(1)(x + 1)$$

Vertex: $(-1, 2)$

Focus: $(0, 2)$

Directrix: $x = -2$



25. $x^2 + 4x + 6y - 2 = 0$

$$x^2 + 4x = -6y + 2$$

$$x^2 + 4x + 4 = -6y + 2 + 4$$

$$(x + 2)^2 = -6(y - 1)$$

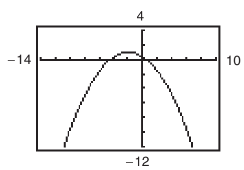
$$(x + 2)^2 = 4\left(-\tfrac{3}{2}\right)(y - 1)$$

$h = -2, k = 1, p = -\frac{3}{2}$

Vertex: $(-2, 1)$

Focus: $\left(-2, -\frac{1}{2}\right)$

Directrix: $y = \frac{5}{2}$



On a graphing calculator, enter:

$y_1 = -\frac{1}{6}(x^2 + 4x - 2)$

26. $x^2 - 2x + 8y + 9 = 0$

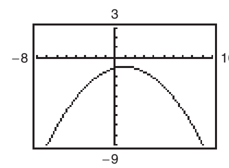
$$x^2 - 2x + 1 = -8y - 9 + 1$$

$$(x - 1)^2 = -8(y + 1) = 4(-2)(y + 1)$$

Vertex: $(1, -1)$

Focus: $(1, -3)$

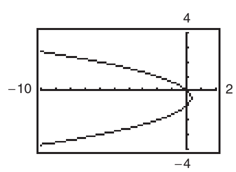
Directrix: $y = 1$



27. $y^2 + x + y = 0$

$y^2 + y + \frac{1}{4} = -x + \frac{1}{4}$

$\left(y + \frac{1}{2}\right)^2 = 4\left(-\frac{1}{4}\right)\left(x - \frac{1}{4}\right)$



$h = \frac{1}{4}, k = -\frac{1}{2}, p = -\frac{1}{4}$

Vertex: $\left(\frac{1}{4}, -\frac{1}{2}\right)$

Focus: $\left(0, -\frac{1}{2}\right)$

Directrix: $x = \frac{1}{2}$

To use a graphing calculator, enter:

$y_1 = -\frac{1}{2} + \sqrt{\frac{1}{4} - x}$

$y_2 = -\frac{1}{2} - \sqrt{\frac{1}{4} - x}$

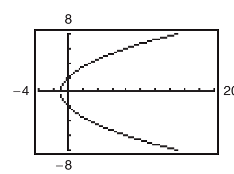
28. $y^2 - 4x - 4 = 0$

$$y^2 = 4x + 4 = 4(1)(x + 1)$$

Vertex: $(-1, 0)$

Focus: $(0, 0)$

Directrix: $x = -2$



29. Vertex: $(0, 0) \implies h = 0, k = 0$

Graph opens upward.

$x^2 = 4py$

Point on graph: $(3, 6)$

$3^2 = 4p(6)$

$9 = 24p$

$\frac{3}{8} = p$

Thus, $x^2 = 4\left(\frac{3}{8}\right)y \implies x^2 = \frac{3}{2}y$.

30. Point: $(-2, 6)$

$$x = ay^2$$

$$-2 = a(6)^2$$

$$-\frac{1}{18} = a$$

$$x = -\frac{1}{18}y^2$$

$$y^2 = -18x$$

31. Vertex: $(0, 0) \implies h = 0, k = 0$

Focus: $\left(0, -\frac{3}{2}\right) \implies p = -\frac{3}{2}$

$x^2 = 4py$

$x^2 = 4\left(-\frac{3}{2}\right)y$

$x^2 = -6y$

32. Focus: $\left(\frac{5}{2}, 0\right) \Rightarrow p = \frac{5}{2}$

$y^2 = 4px$

$y^2 = 10x$

33. Vertex: $(0, 0) \Rightarrow h = 0, k = 0$

Focus: $(-2, 0) \Rightarrow p = -2$

$y^2 = 4px$

$y^2 = 4(-2)x$

$y^2 = -8x$

34. Focus: $(0, -2) \Rightarrow p = -2$

$x^2 = 4py$

$x^2 = -8y$

35. Vertex: $(0, 0) \Rightarrow h = 0, k = 0$

Directrix: $y = -1 \Rightarrow p = 1$

$x^2 = 4py$

$x^2 = 4(1)y$

$x^2 = 4y$

36. Directrix: $y = 3 \Rightarrow p = -3$

$x^2 = 4py$

$x^2 = -12y$

37. Vertex: $(0, 0) \Rightarrow h = 0, k = 0$

Directrix: $x = 2 \Rightarrow p = -2$

$y^2 = 4px$

$y^2 = 4(-2)x$

$y^2 = -8x$

38. Directrix: $x = -3 \Rightarrow p = 3$

$y^2 = 4px$

$y^2 = 12x$

39. Vertex: $(0, 0) \Rightarrow h = 0, k = 0$

Horizontal axis and passes through the point (4, 6)

$y^2 = 4px$

$6^2 = 4p(4)$

$36 = 16p \Rightarrow p = \frac{9}{4}$

$y^2 = 4\left(\frac{9}{4}\right)x$

$y^2 = 9x$

40. Vertical axis

Passes through: $(-3, -3)$

$x^2 = 4py$

$(-3)^2 = 4p(-3)$

$9 = -12p$

$p = -\frac{3}{4}$

$x^2 = -3y$

41. Vertex: (3, 1) and opens downward. Passes through (2, 0) and (4, 0).

$$\begin{aligned} y &= -(x-2)(x-4) \\ &= -x^2 + 6x - 8 \\ &= -(x-3)^2 + 1 \end{aligned}$$

$(x - 3)^2 = -(y - 1)$

42. Vertex: $(5, 3) \Rightarrow h = 5, k = 3$

Passes through: (4.5, 4)

$(y - k)^2 = 4p(x - h)$

$(y - 3)^2 = 4p(x - 5)$

$1 = 4p(4.5 - 5)$

$p = -\frac{1}{2}$

$(y - 3)^2 = -2(x - 5)$

43. Vertex: $(-4, 0)$ and opens to the right.

Passes through (0, 4).

$(y - 0)^2 = 4p(x + 4)$

$4^2 = 4p(0 + 4)$

$16 = 16p$

$1 = p$

$y^2 = 4(x + 4)$

44. Vertex: $(3, -3) \Rightarrow h = 3, k = -3$

Passes through: (0, 0)

$(x - h)^2 = 4p(y - k)$

$(x - 3)^2 = 4p(y + 3)$

$(0 - 3)^2 = 4p(0 + 3)$

$9 = 12p$

$p = \frac{3}{4}$

$(x - 3)^2 = 3(y + 3)$

45. Vertex: (5, 2)

Focus: (3, 2)

Horizontal axis

$p = 3 - 5 = -2$

$(y - 2)^2 = 4(-2)(x - 5)$

$(y - 2)^2 = -8(x - 5)$

46. Vertex: $(-1, 2) \Rightarrow h = -1, k = 2$

Focus: $(-1, 0) \Rightarrow p = -2$

$(x - h)^2 = 4p(y - k)$

$(x + 1)^2 = 4(-2)(y - 2)$

$(x + 1)^2 = -8(y - 2)$

47. Vertex: $(0, 4)$

Directrix: $y = 2$

Vertical axis

$$p = 4 - 2 = 2$$
$$(x - 0)^2 = 4(2)(y - 4)$$
$$x^2 = 8(y - 4)$$

48. Vertex: $(-2, 1) \Rightarrow h = -2, k = 1$

Directrix: $x = 1 \Rightarrow p = -3$

$$(y - k)^2 = 4p(x - h)$$
$$(y - 1)^2 = 4(-3)(x + 2)$$
$$(y - 1)^2 = -12(x + 2)$$

49. Focus: $(2, 2)$

Directrix: $x = -2$

Horizontal axis

Vertex: $(0, 2)$

$$p = 2 - 0 = 2$$
$$(y - 2)^2 = 4(2)(x - 0)$$
$$(y - 2)^2 = 8x$$

50. Focus: $(0, 0)$

Directrix: $y = 8 \Rightarrow p = -4$

$$\Rightarrow h = 0, k = 4$$
$$(x - h)^2 = 4p(y - k)$$
$$x^2 = 4(-4)(y - 4)$$
$$x^2 = -16(y - 4)$$

51. $(y - 3)^2 = 6(x + 1)$

For the upper half of the parabola:

$$y - 3 = \sqrt{6(x + 1)}$$
$$y = \sqrt{6(x + 1)} + 3$$

52. $(y + 1)^2 = 2(x - 4)$

$$y + 1 = \pm\sqrt{2(x - 4)}$$
$$y = -1 \pm \sqrt{2(x - 4)}$$

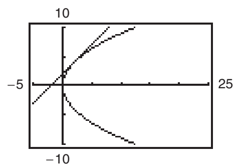
Lower half of parabola:

$y = -1 - \sqrt{2(x - 4)}$

53. $y^2 - 8x = 0 \Rightarrow y = \pm\sqrt{8x}$

$x - y + 2 = 0 \Rightarrow y = x + 2$

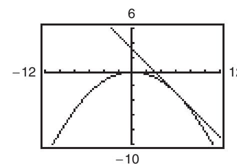
The point of tangency is $(2, 4)$.



54. $x^2 + 12y = 0 \Rightarrow y_1 = -\frac{1}{12}x^2$

$x + y - 3 = 0 \Rightarrow y_2 = 3 - x$

Using the trace or intersect feature, the point of tangency is $(6, -3)$.



55. $x^2 = 2y \Rightarrow p = \dfrac{1}{2}$

Point: $(4, 8)$

Focus: $\left(0, \dfrac{1}{2}\right)$

$$d_1 = \frac{1}{2} - b$$
$$d_2 = \sqrt{(4 - 0)^2 + \left(8 - \frac{1}{2}\right)^2}$$
$$= \frac{17}{2}$$
$$d_1 = d_2 \Rightarrow b = -8$$

Slope: $m = \dfrac{8 - (-8)}{4 - 0} = 4$

$$y = 4x - 8 \Rightarrow 0 = 4x - y - 8$$

x-intercept: $(2, 0)$

56.

$$x^2 = 2y$$
$$x^2 = 4\left(\frac{1}{2}\right)y$$
$$4\left(\frac{1}{2}\right)y = x^2$$
$$p = \frac{1}{2}$$

Focus: $\left(0, \dfrac{1}{2}\right)$

$$d_1 = \frac{1}{2} - b$$
$$d_2 = \sqrt{(-3 - 0)^2 + \left(\frac{9}{2} - \frac{1}{2}\right)^2} = 5$$
$$\frac{1}{2} - b = 5$$
$$b = -\frac{9}{2}$$
$$m = \frac{-(9/2) - (9/2)}{0 + 3} = -3$$

Tangent line: $y = -3x - \dfrac{9}{2} \Rightarrow 6x + 2y + 9 = 0$

x-intercept: $\left(-\dfrac{3}{2}, 0\right)$

57. $y = -2x^2 \implies x^2 = -\frac{1}{2}y \implies p = -\frac{1}{8}$

Point: $(-1, -2)$

Focus: $\left(0, -\frac{1}{8}\right)$

$d_1 = b - \left(-\frac{1}{8}\right) = b + \frac{1}{8}$

$d_2 = \sqrt{(-1-0)^2 + \left(-2 - \left(-\frac{1}{8}\right)\right)^2}$

$= \frac{17}{8}$

$d_1 = d_2 \implies b = 2$

Slope: $m = \dfrac{-2-2}{-1-0} = 4$

$y = 4x + 2 \implies 0 = 4x - y + 2$

x-intercept: $\left(-\frac{1}{2}, 0\right)$

58.
$$y = -2x^2$$
$$-\frac{1}{2}y = x^2$$
$$4\left(-\frac{1}{8}\right)y = x^2$$
$$p = -\frac{1}{8}$$

Focus: $\left(0, -\frac{1}{8}\right)$

$d_1 = \frac{1}{8} + b$

$d^2 = \sqrt{(2-0)^2 + \left(-8 - \left(-\frac{1}{8}\right)\right)^2} = \frac{65}{8}$

$\frac{1}{8} + b = \frac{65}{8}$

$b = \frac{64}{8} = 8$

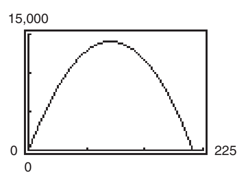
$m = \dfrac{-8-8}{2-0} = -8$

Tangent line: $y = -8x + 8 \implies 8x + y - 8 = 0$

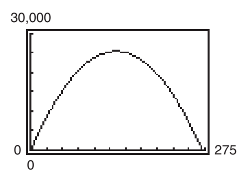
x-intercept: $(1, 0)$

59.
$$(x - 106)^2 = -\tfrac{4}{5}(R - 14{,}045)$$
$$x^2 - 212x + 11{,}236 = -\tfrac{4}{5}R + 11{,}236$$
$$R = 265x - \tfrac{5}{4}x^2$$

The revenue is maximum when $x = 106$ units.



60. Maximum revenue occurs at $x = 135$.



61. Vertex: $(0, 0) \implies h = 0, k = 0$

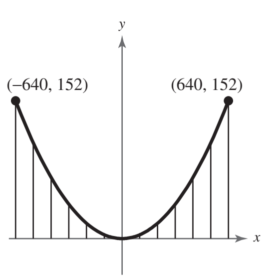
Focus: $(0, 4.5) \implies p = 4.5$

$(x - h)^2 = 4p(y - k)$

$(x - 0)^2 = 4(4.5)(y - 0)$

$x^2 = 18y$ or $y = \frac{1}{18}x^2$

62. (a)



(b) Vertex: $(0, 0)$; opens upward

$y - 0 = a(x - 0)^2$

$152 = a(640)^2$

$\dfrac{152}{640^2} = a$

$\dfrac{19}{51{,}200} = a$

An equation of the cables is

$y = \dfrac{19}{51{,}200}x^2.$

(c)

Distance, x	Height, y
0	0
250	23.19
400	59.38
500	92.77
1000	371.09

63. (a) Vertex: $(0, 0) \implies h = 0, k = 0$

Points on the parabola: $(\pm 16, -0.4)$

$$x^2 = 4py$$
$$(\pm 16)^2 = 4p(-0.4)$$
$$256 = -1.6p$$
$$-160 = p$$
$$x^2 = 4(-160y)$$
$$x^2 = -640y$$
$$y = -\tfrac{1}{640}x^2$$

(b) When $y = -0.1$ we have

$$-0.1 = -\tfrac{1}{640}x^2$$
$$64 = x^2$$
$$\pm 8 = x.$$

Thus, 8 feet away from the center of the road, the road surface is 0.1 foot lower than in the middle.

64. Vertex: $(0, 0)$

$$(y - 0)^2 = 4p(x - 0)$$
$$y^2 = 4px$$

At $(1000, 800)$: $800^2 = 4p(1000) \implies p = 160$

$$y^2 = 4(160)x$$
$$y^2 = 640x$$

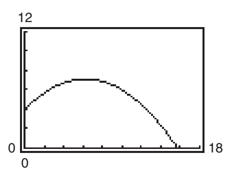
65. (a) $V = 17{,}500\sqrt{2}$ mi/hr

$\approx 24{,}750$ mi/hr

(b) $p = -4100, (h, k) = (0, 4100)$

$$(x - 0)^2 = 4(-4100)(y - 4100)$$
$$x^2 = -16{,}400(y - 4100)$$

66. (a)



(b) Highest point: $(6.25, 7.125)$

Range: 15.69 feet

67. (a) $x^2 = -\dfrac{(32)^2}{16}(y - 75)$

$$x^2 = -64(y - 75)$$

(b) When $y = 0$, $x^2 = -64(-75) = 4800$.

Thus, $x = \sqrt{4800} = 40\sqrt{3} \approx 69.3$ feet.

68. $\dfrac{540 \text{ mi}}{1 \text{ hr}} \cdot \dfrac{5280 \text{ ft}}{1 \text{ mi}} \cdot \dfrac{1 \text{ hr}}{60 \text{ min}} \cdot \dfrac{1 \text{ min}}{60 \text{ s}} = 792$ ft/s

$s = 30{,}000$

The crate hits the ground when $y = 0$.

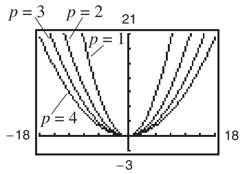
$$x^2 = \frac{-v^2}{16}(y - s)$$
$$x^2 = -\frac{(792)^2}{16}(0 - 30{,}000)$$
$$x^2 = 1{,}176{,}120{,}000$$
$$x \approx 34{,}295$$

The distance is about 34,295 feet.

69. False. It is not possible for a parabola to intersect its directrix. If the graph crossed the directrix there would exist points closer to the directrix than the focus.

70. True. If the axis (line connecting the vertex and focus) is horizontal, then the directrix must be vertical.

71. (a)



As p increases, the graph becomes wider.

(b) $(0, 1), (0, 2), (0, 3), (0, 4)$

(c) 4, 8, 12, 16. The chord passing through the focus and parallel to the directrix has length $|4p|$.

(d) This provides an easy way to determine two additional points on the graph, each of which is $|2p|$ units away from the focus on the chord.

72. (a) $A = \frac{8}{3}(2)^{1/2}(4)^{3/2} = \frac{8}{3}(\sqrt{2})(8) = \frac{64\sqrt{2}}{3}$ square units

(b) As p approaches zero, the parabola becomes narrower and narrower, thus the area becomes smaller and smaller.

73. $y - y_1 = \frac{x_1}{2p}(x - x_1)$

Slope: $m = \frac{x_1}{2p}$

74. *Sample answer*: Any light ray (or other electromagnetic radiation) that enters a parabolic reflector (a surface for which any cross section containing the axis is a parabola) in a direction parallel to the axis of the surface will be reflected to the focus of the surface (the focus of any of the cross-sectional parabolas). Conversely, any ray projected from the focus in a direction that intersects the parabolic surface will be reflected in a direction parallel to the axis.

75. $f(x) = x^3 - 2x^2 + 2x - 4$

Possible rational zeros: $\pm 1, \pm 2, \pm 4$

76. $f(x) = 2x^3 + 4x^2 - 3x + 10$

Rational zeros $\frac{p}{q}$: p = factor of 10, q = factor of 2

Possible rational zeros: $\pm\frac{1}{2}, \pm 1, \pm 2, \pm\frac{5}{2}, \pm 5, \pm 10$

77. $f(x) = 2x^5 + x^2 + 16$

Possible rational zeros: $\pm 1, \pm 2, \pm 4, \pm 8, \pm 16, \pm\frac{1}{2}$

78. $f(x) = 3x^3 - 12x + 22$

Rational zeros $\frac{p}{q}$: p = factor of 22, q = factor of 3

Possible rational zeros: $\pm\frac{1}{3}, \pm\frac{2}{3}, \pm 1, \pm 2, \pm\frac{11}{3}, \pm\frac{22}{3}, \pm 11, \pm 22$

79.
$$\begin{aligned} f(x) &= (x - 3)[x - (2 + i)][x - (2 - i)] \\ &= (x - 3)[(x - 2) - i][(x - 2) + i] \\ &= (x - 3)(x^2 - 4x + 5) \\ &= x^3 - 7x^2 + 17x - 15 \end{aligned}$$

80. $f(x) = 2x^3 - 3x^2 + 50x - 75$

$$\begin{array}{r|rrrr} \frac{3}{2} & 2 & -3 & 50 & -75 \\ & & 3 & 0 & 75 \\ \hline & 2 & 0 & 50 & 0 \end{array}$$

$2x^2 + 50 = 0 \implies x^2 = -25 \implies x = \pm 5i$

Zeros: $x = \frac{3}{2}, \pm 5i$

81. $g(x) = 6x^4 + 7x^3 - 29x^2 - 28x + 20$

Possible rational roots: $\pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20, \pm\frac{1}{2}, \pm\frac{5}{2}, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{4}{3}, \pm\frac{5}{3}, \pm\frac{10}{3}, \pm\frac{20}{3}, \pm\frac{1}{6}, \pm\frac{5}{6}$

$x = \pm 2$ are both solutions.

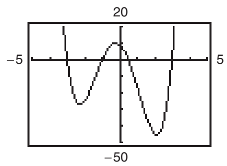
$$\begin{array}{r|rrrrr} 2 & 6 & 7 & -29 & -28 & 20 \\ & & 12 & 38 & 18 & -20 \\ \hline & 6 & 19 & 9 & -10 & 0 \end{array}$$

$$\begin{array}{r|rrrr} -2 & 6 & 19 & 9 & -10 \\ & & -12 & -14 & 10 \\ \hline & 6 & 7 & -5 & 0 \end{array}$$

$$\begin{aligned} g(x) &= (x - 2)(x + 2)(6x^2 + 7x - 5) \\ &= (x - 2)(x + 2)(2x - 1)(3x + 5) \end{aligned}$$

The zeros of $g(x)$ are $x = \pm 2, x = \frac{1}{2}, x = -\frac{5}{3}$.

82. $h(x) = 2x^4 + x^3 - 19x^2 - 9x + 9$



Zeros: $x = \pm 3, -1, \frac{1}{2}$

83. $A = 35°, a = 10, b = 7$

$$\frac{\sin B}{7} = \frac{\sin 35°}{10} \implies \sin B \approx 0.4015 \implies B \approx 23.67°$$

$$C \approx 180° - 35° - 23.67° = 121.33°$$

$$\frac{c}{\sin 121.33°} = \frac{10}{\sin 35°} \implies c \approx 14.89$$

84. $B = 54°, b = 18, c = 11$

Because B is acute and $18 > 11$, one triangle is possible.

$$\sin C = \frac{c \sin B}{b} = \frac{11 \sin 54}{18} \approx 0.49440 \implies C \approx 29.63°$$

$$A = 180° - B - C \approx 180° - 54° - 29.63° = 96.37°$$

$$a = \frac{b}{\sin B}(\sin A) = \frac{18}{\sin 54}(\sin 96.37°) \approx 22.11$$

85. $A = 40°, B = 51°, c = 3$

$$C = 180° - 40° - 51° = 89°$$

$$\frac{a}{\sin 40°} = \frac{3}{\sin 89°} \implies a \approx 1.93$$

$$\frac{b}{\sin 51°} = \frac{3}{\sin 89°} \implies b \approx 2.33$$

86. $B = 26°, C = 104°, a = 19$

$$A = 180° - B - C \approx 180° - 26° - 104° = 50°$$

$$b = \frac{a}{\sin A}(\sin B) = \frac{19}{\sin 50}(\sin 26°) \approx 10.87$$

$$c = \frac{a}{\sin A}(\sin C) = \frac{19}{\sin 50}(\sin 104°) \approx 24.07$$

87. $a = 7, b = 10, c = 16$

$$\cos C = \frac{7^2 + 10^2 - 16^2}{2(7)(10)} \approx -0.7643 \implies C \approx 139.84°$$

$$\frac{\sin B}{10} = \frac{\sin 139.84°}{16} \implies \sin B \approx 0.4031 \implies B \approx 23.77°$$

$$A = 180° - B - C \implies A \approx 16.39°$$

88. $a = 58, b = 28, c = 75$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{784 + 5625 - 3364}{2(28)(75)} = 0.725 \implies A \approx 43.53°$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{3364 + 5625 - 784}{2(58)(75)} = 0.943103 \implies B \approx 19.42°$$

$$C = 180° - A - B \approx 180° - 43.53° - 19.42° = 117.05°$$

89. $A = 65°, b = 5, c = 12$

$$a^2 = 5^2 + 12^2 - 2(5)(12)\cos 65° \implies a \approx 10.8759 \approx 10.88$$

$$\frac{\sin B}{5} = \frac{\sin 65°}{10.8759} \implies \sin B \approx 0.4167 \implies B \approx 24.62°$$

$$C = 180° - A - B \implies C \approx 90.38°$$

90. $B = 71°, a = 21, c = 29$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$b^2 = 441 + 841 - 2(21)(29)\cos 71° \approx 885.458$$

$$b \approx 29.76$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \approx \frac{(29.76)^2 + 841 - 441}{2(29.76)(29)} \approx 0.74484 \implies A \approx 41.85°$$

$$C = 180° - A - B \approx 180° - 41.85° - 71° = 67.15°$$

Section 10.3 Ellipses

- An **ellipse** is the set of all points (x, y) the sum of whose distances from two distinct fixed points **(foci)** is constant.
- The standard equation of an ellipse with center (h, k) and major and minor axes of lengths $2a$ and $2b$ is:
  (a) $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$ if the major axis is horizontal.
  (b) $\dfrac{(x-h)^2}{b^2} + \dfrac{(y-k)^2}{a^2} = 1$ is the major axis is vertical.
- $c^2 = a^2 - b^2$ where c is the distance from the center to a focus.
- The eccentricity of an ellipse is $e = \dfrac{c}{a}$.

Vocabulary Check

1. ellipse; foci
2. major axis, center
3. minor axis
4. eccentricity

1. $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$

Center: $(0, 0)$

$a = 3, b = 2$

Vertical major axis

Matches graph (b).

2. $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

Center: $(0, 0)$

$a = 3, b = 2$

Horizontal major axis

Matches graph (c).

3. $\dfrac{x^2}{4} + \dfrac{y^2}{25} = 1$

Center: $(0, 0)$

$a = 5, b = 2$

Vertical major axis

Matches graph (d).

4. $\dfrac{y^2}{4} + \dfrac{x^2}{4} = 1$

Center: $(0, 0)$

Circle of radius: 2

Matches graph (f).

5. $\dfrac{(x-2)^2}{16} + (y+1)^2 = 1$

Center: $(2, -1)$

$a = 4, b = 1$

Horizontal major axis

Matches graph (a).

6. $\dfrac{(x+2)^2}{9} + \dfrac{(y+2)^2}{4} = 1$

Center: $(-2, -2)$

$a = 3, b = 2$

Horizontal major axis

Matches graph (e).

7. $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$

Ellipse

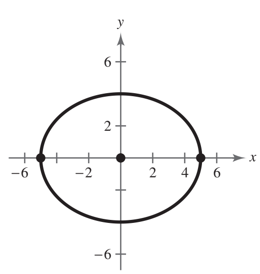
Center: $(0, 0)$

$a = 5, b = 4, c = 3$

Vertices: $(\pm 5, 0)$

Foci: $(\pm 3, 0)$

$e = \dfrac{3}{5}$



8. $\dfrac{x^2}{81} + \dfrac{y^2}{144} = 1$

$a = 12, b = 9,$

$c = \sqrt{63} = 3\sqrt{7}$

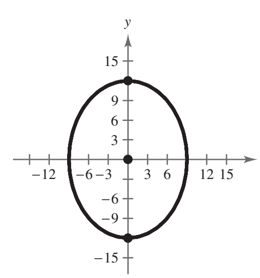
Ellipse

Center: $(0, 0)$

Vertices: $(0, \pm 12)$

Foci: $\left(0, \pm 3\sqrt{7}\right)$

Eccentricity: $e = \dfrac{\sqrt{7}}{4}$

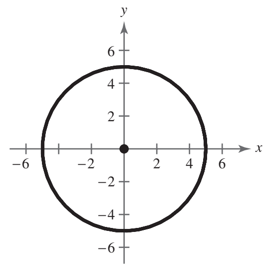


9. $\dfrac{x^2}{25} + \dfrac{y^2}{25} = 1 \implies x^2 + y^2 = 25$

Circle

Center: $(0, 0)$

Radius: 5

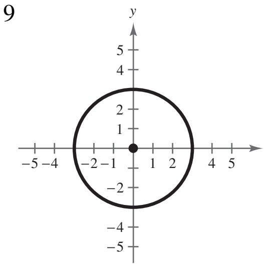


10. $\dfrac{x^2}{9} + \dfrac{y^2}{9} = 1 \implies x^2 + y^2 = 9$

Circle

Center: $(0, 0)$

Radius: 3



11. $\dfrac{x^2}{5} + \dfrac{y^2}{9} = 1$

Ellipse

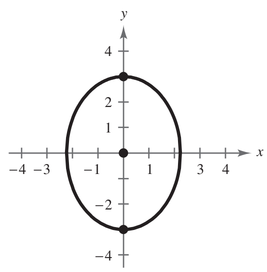
$a = 3, b = \sqrt{5}, c = 2$

Center: $(0, 0)$

Vertices: $(0, \pm 3)$

Foci: $(0, \pm 2)$

$e = \dfrac{2}{3}$



12. $\dfrac{x^2}{64} + \dfrac{y^2}{28} = 1$

$a = 8, b = \sqrt{28} = 2\sqrt{7},$

$c = 6$

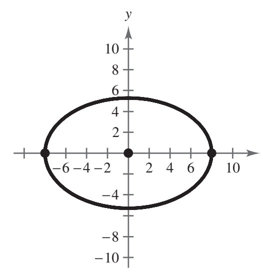
Ellipse

Center: $(0, 0)$

Vertices: $(\pm 8, 0)$

Foci: $(\pm 6, 0)$

Eccentricity: $e = \dfrac{3}{4}$



13. $\dfrac{(x + 3)^2}{16} + \dfrac{(y - 5)^2}{25} = 1$

Ellipse

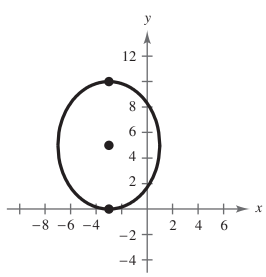
$a = 5, b = 4, c = 3$

Center: $(-3, 5)$

Vertices: $(-3, 10)(-3, 0)$

Foci: $(-3, 8)(-3, 2)$

$e = \dfrac{3}{5}$



14. $\dfrac{(x - 4)^2}{12} + \dfrac{(y + 3)^2}{16} = 1$

$a = 4, b = 2\sqrt{3}, c = 2$

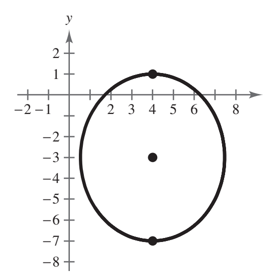
Ellipse

Center: $(4, -3)$

Vertices: $(4, 1), (4, -7)$

Foci: $(4, -1), (4, -5)$

$e = \dfrac{2}{4} = \dfrac{1}{2}$

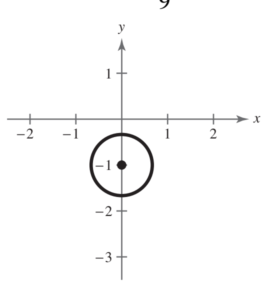


15. $\dfrac{x^2}{4/9} + \dfrac{(y + 1)^2}{4/9} = 1 \implies x^2 + (y + 1)^2 = \dfrac{4}{9}$

Circle

Center: $(0, -1)$

Radius: $\dfrac{2}{3}$



16. $\dfrac{(x + 5)^2}{9/4} + (y - 1)^2 = 1$

Ellipse

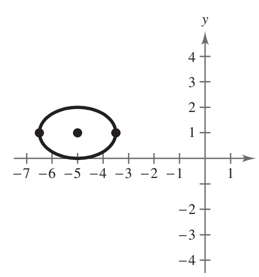
$a = \dfrac{3}{2}, b = 1, c = \dfrac{\sqrt{5}}{2}$

Center: $(-5, 1)$

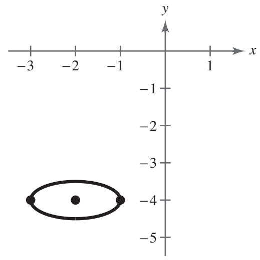
Vertices: $\left(-\dfrac{7}{2}, 1\right), \left(-\dfrac{13}{2}, 1\right)$

Foci: $\left(-5 + \dfrac{\sqrt{5}}{2}, 1\right), \left(-5 - \dfrac{\sqrt{5}}{2}, 1\right)$

$e = \dfrac{\sqrt{5}}{3}$



17. $\dfrac{(x+2)^2}{1} + \dfrac{(y+4)^2}{1/4} = 1$



Ellipse

$a = 1, b = \dfrac{1}{2}, c = \dfrac{\sqrt{3}}{2}$

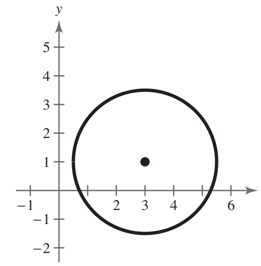
Center: $(-2, -4)$

Vertices: $(-1, -4), (-3, -4)$

Foci: $\left(-2 \pm \dfrac{\sqrt{3}}{2}, -4\right) = \left(\dfrac{-4 \pm \sqrt{3}}{2}, -4\right)$

$e = \dfrac{\sqrt{3}}{2}$

18. $\dfrac{(x-3)^2}{25/4} + \dfrac{(y-1)^2}{25/4} = 1$



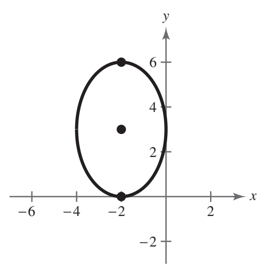
Circle

Center: $(3, 1)$

Radius: $\dfrac{5}{2}$

19.

$$9x^2 + 4y^2 + 36x - 24y + 36 = 0$$
$$9(x^2 + 4x + 4) + 4(y^2 - 6y + 9) = -36 + 36 + 36$$
$$9(x+2)^2 + 4(y-3)^2 = 36$$
$$\frac{(x+2)^2}{4} + \frac{(y-3)^2}{9} = 1$$



Ellipse

$a = 3, b = 2, c = \sqrt{5}$

Center: $(-2, 3)$

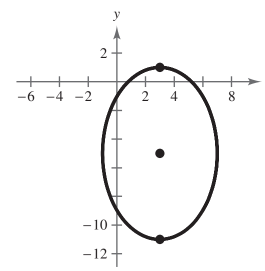
Vertices: $(-2, 6), (-2, 0)$

Foci: $\left(-2, 3 \pm \sqrt{5}\right)$

$e = \dfrac{\sqrt{5}}{3}$

20.

$$9x^2 + 4y^2 - 54x + 40y + 37 = 0$$
$$9(x^2 - 6x + 9) + 4(y^2 + 10y + 25) = -37 + 81 + 100$$
$$\frac{(x-3)^2}{16} + \frac{(y+5)^2}{36} = 1$$



$a = 6, b = 4,$

$c = \sqrt{20} = 2\sqrt{5}$

Ellipse

Center: $(3, -5)$

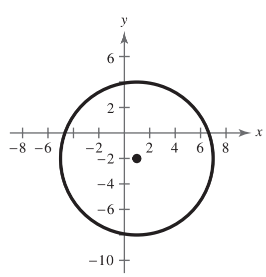
Vertices: $(3, 1), (3, -11)$

Foci: $\left(3, -5 \pm 2\sqrt{5}\right)$

Eccentricity: $e = \dfrac{\sqrt{5}}{3}$

21.

$$x^2 + y^2 - 2x + 4y - 31 = 0$$
$$(x^2 - 2x + 1) + (y^2 + 4y + 4) = 31 + 1 + 4$$
$$(x-1)^2 + (y+2)^2 = 36$$
$$\frac{(x-1)^2}{36} + \frac{(y+2)^2}{36} = 1$$



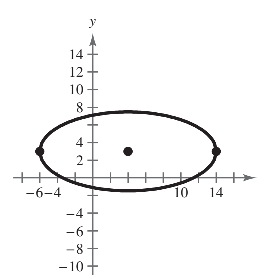
Circle

Center: $(1, -2)$

Radius: 6

22.

$$x^2 + 5y^2 - 8x - 30y - 39 = 0$$
$$(x^2 - 8x + 16) + 5(y^2 - 6y + 9) = 39 + 16 + 45$$
$$(x-4)^2 + 5(y-3)^2 = 100$$
$$\frac{(x-4)^2}{100} + \frac{(y-3)^2}{20} = 1$$



Ellipse

Center: $(4, 3)$

$a = 10, b = \sqrt{20} = 2\sqrt{5},$

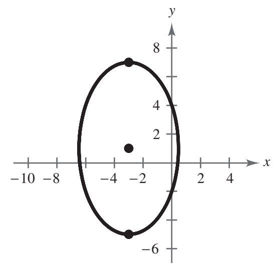
$c = \sqrt{80} = 4\sqrt{5}$

Foci: $\left(4 \pm 4\sqrt{5}, 3\right)$

Vertices: $(14, 3), (-6, 3)$

$e = \dfrac{4\sqrt{5}}{10} = \dfrac{2\sqrt{5}}{5}$

23. $$3x^2 + y^2 + 18x - 2y - 8 = 0$$
$$3(x^2 + 6x + 9) + (y^2 - 2y + 1) = 8 + 27 + 1$$
$$3(x + 3)^2 + (y - 1)^2 = 36$$
$$\frac{(x + 3)^2}{12} + \frac{(y - 1)^2}{36} = 1$$



Ellipse

$a = 6, b = \sqrt{12} = 2\sqrt{3}, c = \sqrt{24} = 2\sqrt{6}$

Center: $(-3, 1)$

Vertices: $(-3, 7), (-3, -5)$

Foci: $\left(-3, 1 \pm 2\sqrt{6}\right)$

Eccentricity: $e = \dfrac{\sqrt{6}}{3}$

24. $$6x^2 + 2y^2 + 18x - 10y + 2 = 0$$
$$6\left(x^2 + 3x + \frac{9}{4}\right) + 2\left(y^2 - 5y + \frac{25}{4}\right) = -2 + \frac{27}{2} + \frac{25}{2}$$
$$6\left(x + \frac{3}{2}\right)^2 + 2\left(y - \frac{5}{2}\right)^2 = 24$$
$$\frac{\left(x + \frac{3}{2}\right)^2}{4} + \frac{\left(y - \frac{5}{2}\right)^2}{12} = 1$$

$a = \sqrt{12} = 2\sqrt{3}, b = 2, c = \sqrt{8} = 2\sqrt{2}$

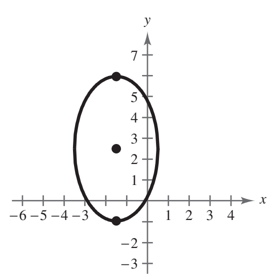
Ellipse

Center: $\left(-\dfrac{3}{2}, \dfrac{5}{2}\right)$

Foci: $\left(-\dfrac{3}{2}, \dfrac{5}{2} \pm 2\sqrt{2}\right)$

Vertices: $\left(-\dfrac{3}{2}, \dfrac{5}{2} \pm 2\sqrt{3}\right)$

$e = \dfrac{2\sqrt{2}}{2\sqrt{3}} = \dfrac{\sqrt{6}}{3}$



25. $$x^2 + 4y^2 - 6x + 20y - 2 = 0$$
$$(x^2 - 6x + 9) + 4\left(y^2 + 5y + \frac{25}{4}\right) = 2 + 9 + 25$$
$$(x - 3)^2 + 4\left(y + \frac{5}{2}\right)^2 = 36$$
$$\frac{(x - 3)^2}{36} + \frac{\left(y + \frac{5}{2}\right)^2}{9} = 1$$

Ellipse

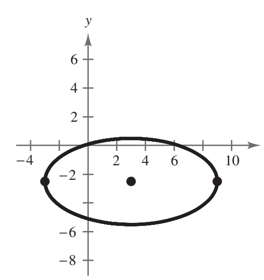
$a = 6, b = 3, c = \sqrt{27} = 3\sqrt{3}$

Center: $\left(3, -\dfrac{5}{2}\right)$

Vertices: $\left(9, -\dfrac{5}{2}\right), \left(-3, -\dfrac{5}{2}\right)$

Foci: $\left(3 \pm 3\sqrt{3}, -\dfrac{5}{2}\right)$

Eccentricity: $e = \dfrac{\sqrt{3}}{2}$

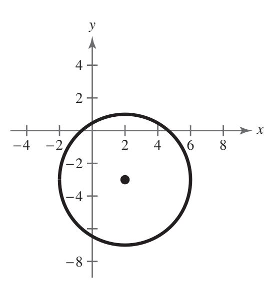


26. $$x^2 + y^2 - 4x + 6y - 3 = 0$$
$$(x^2 - 4x + 4) + (y^2 + 6y + 9) = 3 + 4 + 9$$
$$\frac{(x - 2)^2}{16} + \frac{(y + 3)^2}{16} = 1$$

Circle

Center: $(2, -3)$

Radius: 4

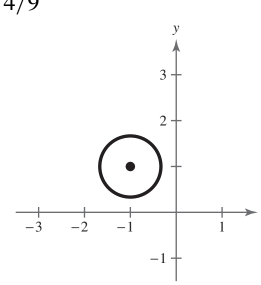


27. $$9x^2 + 9y^2 + 18x - 18y + 14 = 0$$
$$9(x^2 + 2x + 1) + 9(y^2 - 2y + 1) = -14 + 9 + 9$$
$$9(x + 1)^2 + 9(y - 1)^2 = 4$$
$$(x + 1)^2 + (y - 1)^2 = \frac{4}{9}$$
$$\frac{(x + 1)^2}{4/9} + \frac{(y - 1)^2}{4/9} = 1$$

Circle

Center: $(-1, 1)$

Radius: $\dfrac{2}{3}$



28. $16x^2 + 25y^2 - 32x + 50y + 16 = 0$

$$16(x^2 - 2x + 1) + 25(y^2 + 2y + 1) = -16 + 16 + 25$$

$$16(x-1)^2 + 25(y+1)^2 = 25$$

$$\frac{(x-1)^2}{25/16} + (y+1)^2 = 1$$

$a^2 = \frac{25}{16}, b^2 = 1, c^2 = \frac{9}{16}$

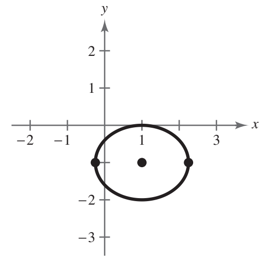
Ellipse

Center: $(1, -1)$

Foci: $\left(\frac{7}{4}, -1\right), \left(\frac{1}{4}, -1\right)$

Vertices: $\left(\frac{9}{4}, -1\right), \left(-\frac{1}{4}, -1\right)$

$e = \frac{3}{5}$



29. $9x^2 + 25y^2 - 36x - 50y + 60 = 0$

$$9(x^2 - 4x + 4) + 25(y^2 - 2y + 1) = -60 + 36 + 25$$

$$9(x-2)^2 + 25(y-1)^2 = 1$$

$$\frac{(x-2)^2}{1/9} + \frac{(y-1)^2}{1/25} = 1$$

Ellipse

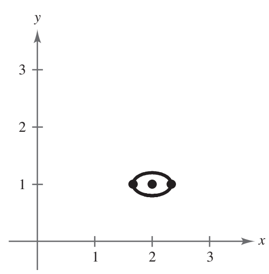
$a = \frac{1}{3}, b = \frac{1}{5}, c = \frac{4}{15}$

Center: $(2, 1)$

Vertices: $\left(\frac{5}{3}, 1\right), \left(\frac{7}{3}, 1\right)$

Foci: $\left(\frac{34}{15}, 1\right), \left(\frac{26}{15}, 1\right)$

Eccentricity: $e = \frac{4}{5}$



30. $16x^2 + 16y^2 - 64x + 32y + 55 = 0$

$$16(x^2 - 4x + 4) + 16(y^2 + 2y + 1) = -55 + 64 + 16$$

$$16(x-2)^2 + 16(y+1)^2 = 25$$

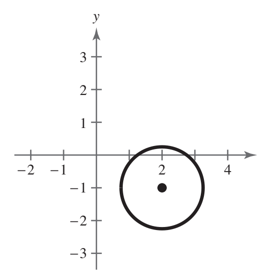
$$(x-2)^2 + (y+1)^2 = \frac{25}{16}$$

$$\frac{(x-2)^2}{25/16} + \frac{(y+1)^2}{25/16} = 1$$

Circle

Center: $(2, -1)$

Radius: $\frac{5}{4}$



31. $5x^2 + 3y^2 = 15$

$$\frac{x^2}{3} + \frac{y^2}{5} = 1$$

Center: $(0, 0)$

$a = \sqrt{5}, b = \sqrt{3}, c = \sqrt{2}$

Foci: $(0, \pm\sqrt{2})$

Vertices: $(0, \pm\sqrt{5})$

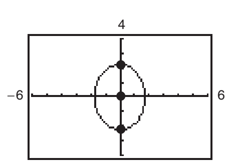
$e = \frac{\sqrt{10}}{5}$

To graph, solve for y.

$$y^2 = \frac{15 - 5x^2}{3}$$

$$y_1 = \sqrt{\frac{15 - 5x^2}{3}}$$

$$y_2 = -\sqrt{\frac{15 - 5x^2}{3}}$$



32. $3x^2 + 4y^2 = 12$

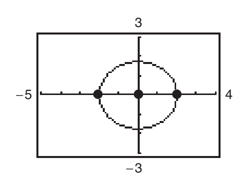
$$\frac{x^2}{4} + \frac{y^2}{3} = 1$$

$a^2 = 4, b^2 = 3, c^2 = 1$

Center: $(0, 0)$

Vertices: $(\pm 2, 0)$

Foci: $(\pm 1, 0)$



33. $12x^2 + 20y^2 - 12x + 40y - 37 = 0$

$$12\left(x^2 - x + \frac{1}{4}\right) + 20(y^2 + 2y + 1) = 37 + 3 + 20$$

$$12\left(x - \frac{1}{2}\right)^2 + 20(y + 1)^2 = 60$$

$$\frac{\left(x - \frac{1}{2}\right)^2}{5} + \frac{(y + 1)^2}{3} = 1$$

$a = \sqrt{5}, b = \sqrt{3}, c = \sqrt{2}$

Center: $\left(\frac{1}{2}, -1\right)$

Foci: $\left(\frac{1}{2} \pm \sqrt{2}, -1\right)$

Vertices: $\left(\frac{1}{2} \pm \sqrt{5}, -1\right)$

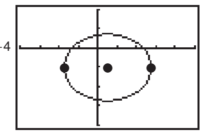
$e = \dfrac{\sqrt{10}}{5}$

To graph, solve for y.

$$(y + 1)^2 = 3\left[1 - \frac{(x - 0.5)^2}{5}\right]$$

$$y_1 = -1 + \sqrt{3\left[1 - \frac{(x - 0.5)^2}{5}\right]}$$

$$y_2 = -1 - \sqrt{3\left[1 - \frac{(x - 0.5)^2}{5}\right]}$$



34. $36x^2 + 9y^2 + 48x - 36y - 72 = 0$

$$36\left(x^2 + \frac{4}{3}x + \frac{4}{9}\right) + 9(y^2 - 4y + 4) = 72 + 16 + 36$$

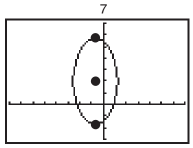
$$\frac{\left(x + \frac{2}{3}\right)^2}{\frac{31}{9}} + \frac{(y - 2)^2}{\frac{124}{9}} = 1$$

$a^2 = \dfrac{124}{9}, b^2 = \dfrac{31}{9}, c^2 = \dfrac{31}{3}$

Center: $\left(-\frac{2}{3}, 2\right)$

Vertices: $\left(-\frac{2}{3}, 2 \pm \frac{2\sqrt{31}}{3}\right)$

Foci: $\left(-\frac{2}{3}, 2 \pm \frac{\sqrt{93}}{3}\right)$



35. Center: $(0, 0)$

$a = 4, b = 2$

Vertical major axis

$$\frac{(x - h)^2}{b^2} + \frac{(y - k)^2}{a^2} = 1$$

$$\frac{x^2}{4} + \frac{y^2}{16} = 1$$

36. Vertices: $(\pm 2, 0) \implies a = 2$

Endpoints of minor axis:

$\left(0, \pm\frac{3}{2}\right) \implies b = \frac{3}{2}$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{2^2} + \frac{y^2}{(3/2)^2} = 1$$

$$\frac{x^2}{4} + \frac{4y^2}{9} = 1$$

37. Vertices: $(\pm 6, 0)$

$a = 6, c = 2 \implies b = \sqrt{32} = 4\sqrt{2}$

Foci: $(\pm 2, 0)$

Horizontal major axis

Center: $(0, 0)$

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$$

$$\frac{x^2}{36} + \frac{y^2}{32} = 1$$

38. Vertices: $(0, \pm 8) \implies a = 8$

Foci: $(0, \pm 4) \implies c = 4$

$b^2 = a^2 - c^2 = 64 - 16 = 48$

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

$$\frac{x^2}{48} + \frac{y^2}{64} = 1$$

39. Foci: $(\pm 5, 0) \implies c = 5$

Center: $(0, 0)$

Horizontal major axis

Major axis of length 12 $\implies 2a = 12$

$$a = 6$$

$6^2 - b^2 = 5^2 \implies b^2 = 11$

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$$\frac{x^2}{36} + \frac{y^2}{11} = 1$$

40. Foci: $(\pm 2, 0) \implies c = 2$

Major axis length: $8 \implies a = 4$

$b^2 = a^2 - c^2 = 16 - 4 = 12$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{16} + \frac{y^2}{12} = 1$$

41. Vertices: $(0, \pm 5) \implies a = 5$

Center: $(0, 0)$

Vertical major axis

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

$$\frac{x^2}{b^2} + \frac{y^2}{25} = 1$$

Point: $(4, 2)$

$$\frac{4^2}{b^2} + \frac{2^2}{25} = 1$$

$$\frac{16}{b^2} = 1 - \frac{4}{25} = \frac{21}{25}$$

$$400 = 21b^2$$

$$\frac{400}{21} = b^2$$

$$\frac{x^2}{400/21} + \frac{y^2}{25} = 1$$

$$\frac{21x^2}{400} + \frac{y^2}{25} = 1$$

42. Major axis vertical

Passes through: $(0, 4)$ and $(2, 0)$

$a = 4, b = 2$

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

$$\frac{x^2}{4} + \frac{y^2}{16} = 1$$

43. Center: $(2, 3)$

$a = 3, b = 1$

Vertical major axis

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

$$\frac{(x-2)^2}{1} + \frac{(y-3)^2}{9} = 1$$

44. Vertices: $(4, \pm 4) \implies a = 4$

Center: $(4, 0) \implies h = 4, k = 0$

Endpoints of minor axis: $(1, 0), (7, 0) \implies b = 3$

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

$$\frac{(x-4)^2}{9} + \frac{y^2}{16} = 1$$

45. Center: $(-2, 3)$

$a = 4, b = 3$

Horizontal major axis

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$$\frac{(x+2)^2}{16} + \frac{(y-3)^2}{9} = 1$$

46. Vertices: $(0, -1), (4, -1) \Rightarrow a = 2$

Center: $(2, -1) \Rightarrow h = 2, k = -1$

Endpoints of minor axis: $(2, 0), (2, -2) \Rightarrow b = 1$

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$$\frac{(x-2)^2}{4} + \frac{(y+1)^2}{1} = 1$$

47. Vertices: $(0, 4), (4, 4) \Rightarrow a = 2$

Minor axis of length 2 $\Rightarrow b = 1$

Center: $(2, 4) = (h, k)$

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$$\frac{(x-2)^2}{4} + \frac{(y-4)^2}{1} = 1$$

48. Foci: $(0, 0), (4, 0) \Rightarrow c = 2, h = 2, k = 0$

Major axis length: $8 \Rightarrow a = 4$

$b^2 = a^2 - c^2 = 16 - 4 = 12$

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$$\frac{(x-2)^2}{16} + \frac{y^2}{12} = 1$$

49. Foci: $(0, 0), (0, 8) \Rightarrow c = 4$

Major axis of length 16 $\Rightarrow a = 8$

$b^2 = a^2 - c^2 = 64 - 16 = 48$

Center:$(0, 4) = (h, k)$

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

$$\frac{x^2}{48} + \frac{(y-4)^2}{64} = 1$$

50. Center: $(2, -1) \Rightarrow h = 2, k = -1$

Vertex: $\left(2, \frac{1}{2}\right) \Rightarrow a = \frac{3}{2}$

Minor axis length: $2 \Rightarrow b = 1$

$$\frac{(x-h)}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

$$\frac{(x-2)^2}{1} + \frac{(y+1)^2}{\left(\frac{3}{2}\right)^2} = 1$$

$$(x-2)^2 + \frac{4(y+1)^2}{9} = 1$$

51. Center: $(0, 4)$

Vertices: $(-4, 4), (4, 4) \Rightarrow a = 4$

$a = 2c \Rightarrow 4 = 2c \Rightarrow c = 2$

$2^2 = 4^2 - b^2 \Rightarrow b^2 = 12$

Horizontal major axis

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$$\frac{x^2}{16} + \frac{(y-4)^2}{12} = 1$$

52. Center: $(3, 2) \Rightarrow h = 3, k = 2$

$a = 3c$

Foci: $(1, 2), (5, 2) \Rightarrow c = 2, a = 6$

$b^2 = a^2 - c^2 = 36 - 4 = 32$

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$$\frac{(x-3)^2}{36} + \frac{(y-2)^2}{32} = 1$$

53. Vertices: $(0, 2), (4, 2) \Rightarrow a = 2$

Center: $(2, 2)$

Endpoints of the minor axis: $(2, 3), (2, 1) \Rightarrow b = 1$

Horizontal major axis

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$$\frac{(x-2)^2}{4} + \frac{(y-2)^2}{1} = 1$$

54. Vertices: $(5, 0), (5, 12) \Rightarrow a = 6$

Endpoints of the minor axis:

$(1, 6), (9, 6) \Rightarrow b = 4$

Center: $(5, 6) \Rightarrow h = 5, k = 6$

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

$$\frac{(x-5)^2}{16} + \frac{(y-6)^2}{36} = 1$$

55. Vertices: $(\pm 5, 0) \Rightarrow a = 5$

Eccentricity: $\frac{3}{5} \Rightarrow c = \frac{3}{5}a = 3$

$b^2 = a^2 - c^2 = 25 - 9 = 16$

Center: $(0, 0) = (h, k)$

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$$\frac{x^2}{25} + \frac{y^2}{16} = 1$$

56. Vertices: $(0, \pm 8) \Rightarrow a = 8$

$e = \frac{1}{2} \Rightarrow \frac{c}{a} = \frac{1}{2}, c = 4$

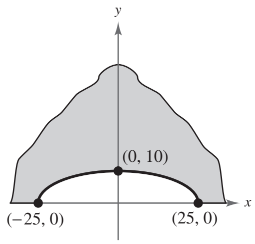
$b^2 = a^2 - c^2 = 64 - 16 = 48$

Center: $(0, 0)$

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

$$\frac{x^2}{48} + \frac{y^2}{64} = 1$$

57. (a)



(b) $a = 25, b = 10$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{625} + \frac{y^2}{100} = 1$$

(c) When $x = \pm 4$:

$$\frac{4^2}{625} + \frac{y^2}{100} = 1$$

$$y^2 = 100\left(1 - \frac{16}{625}\right) = \frac{2436}{25}$$

$$y = \sqrt{\frac{2436}{25}} \approx 9.87 \text{ feet} > 9 \text{ feet}$$

Yes. If the truck travels down the center of the tunnel, it will clear the opening of the arch.

58. The tacks should be placed at the foci and the length of the string is the length of the major axis, $2a$.

Center: $(0, 0)$

$a = 3, b = 2, c = \sqrt{5}$

Foci (Positions of the tacks): $\left(\pm\sqrt{5}, 0\right)$

Length of string: 6 feet

59. (a) $a = \dfrac{35.88}{2} = 17.94$

$$e = \frac{c}{a} = 0.967$$

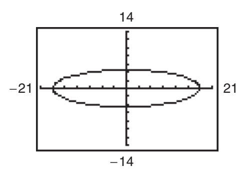
$$c = ea \approx 17.35$$

$$b^2 = a^2 - c^2 \approx 20.82$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{321.84} + \frac{y^2}{20.82} = 1$$

(b)



(c) The sun's center is at a focus of the orbit, 17.35 astronomical units from the center of the orbit.

Apogee $\approx 17.35 + \frac{1}{2}(35.88) = 35.29$ astronomical units

Perigee $\approx \frac{1}{2}(35.88) - 17.35 + \ = 0.59$ astronomical units

60. $a + c = 6378 + 947 = 7325$

$a - c = 6378 + 228 = 6606$

Solving this system for a and c yields $a = 6965.5$ and $c = 359.5$.

$$e = \frac{c}{a} = \frac{359.5}{6965.5} \approx 0.052$$

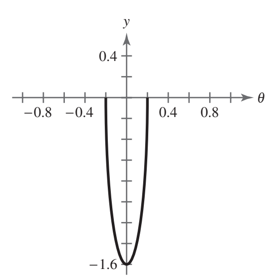
61. (a) The equation is the bottom half of the ellipse.

$$\frac{\theta^2}{(0.2)^2} + \frac{y^2}{(1.6)^2} = 1$$

$$y = -1.6\sqrt{1 - \frac{\theta^2}{0.04}}$$

$$= -8\sqrt{0.04 - \theta^2}$$

(b)



(c) The bottom half models the motion of the pendulum.

62. For $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we have $c^2 = a^2 - b^2$.

When $x = c$: $\frac{c^2}{a^2} + \frac{y^2}{b^2} = 1$

$$y^2 = b^2\left(1 - \frac{a^2 - b^2}{a^2}\right) = \frac{b^4}{a^2}$$

$$y = \frac{b^2}{a}$$

Length of latus rectum: $2y = \frac{2b^2}{a}$

63. $\frac{x^2}{9} + \frac{y^2}{16} = 1$

$a = 4, b = 3, c = \sqrt{7}$

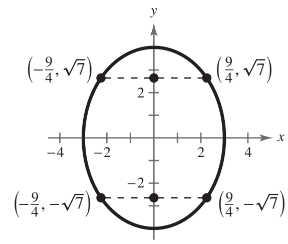
Points on the ellipse:

$(\pm 3, 0), (0, \pm 4)$

Length of latus recta:

$\frac{2b^2}{a} = \frac{2(3)^2}{4} = \frac{9}{2}$

Additional points: $\left(\pm\frac{9}{4}, -\sqrt{7}\right), \left(\pm\frac{9}{4}, \sqrt{7}\right)$



64. $\frac{x^2}{4} + \frac{y^2}{1} = 1$

$a = 2, b = 1, c = \sqrt{3}$

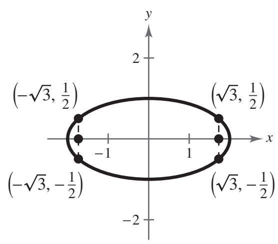
Points on the ellipse:

$(\pm 2, 0), (0, \pm 1)$

Length of latera recta:

$\frac{2b^2}{a} = \frac{2(1)^2}{2} = 1$

Additional points: $\left(-\sqrt{3}, \pm\frac{1}{2}\right), \left(\sqrt{3}, \pm\frac{1}{2}\right)$



65. $5x^2 + 3y^2 = 15$

$\frac{x^2}{3} + \frac{y^2}{5} = 1$

$a = \sqrt{5}, b = \sqrt{3}, c = \sqrt{2}$

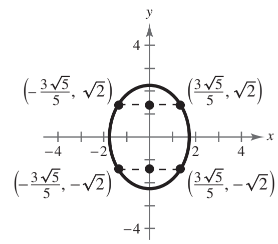
Points on the ellipse:

$\left(\pm\sqrt{3}, 0\right), \left(0, \pm\sqrt{5}\right)$

Length of latus recta:

$\frac{2b^2}{a} = \frac{2 \cdot 3}{\sqrt{5}} = \frac{6\sqrt{5}}{5}$

Additional points: $\left(\pm\frac{3\sqrt{5}}{5}, \pm\sqrt{2}\right)$



66. $9x^2 + 4y^2 = 36$

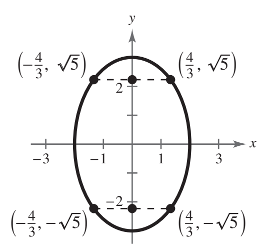
$\frac{x^2}{4} + \frac{y^2}{9} = 1$

$a = 3, b = 2, c = \sqrt{5}$

Points on the ellipse: $(\pm 2, 0), (0, \pm 3)$

Length of latera recta: $\frac{2b^2}{a} = \frac{2 \cdot 2^2}{3} = \frac{8}{3}$

Additional points: $\left(\pm\frac{4}{3}, -\sqrt{5}\right), \left(\pm\frac{4}{3}, \sqrt{5}\right)$



67. False. The graph of $\frac{x^2}{4} + y^4 = 1$ is not an ellipse. The degree on y is 4, not 2.

68. True. If e is close to 1, the ellipse is elongated and the foci are close to the vertices.

69. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(a) $a + b = 20 \Rightarrow b = 20 - a$

$A = \pi ab = \pi a(20 - a)$

(b) $264 = \pi a(20 - a)$

$$0 = -\pi a^2 + 20\pi a - 264$$

$$0 = \pi a^2 - 20\pi a + 264$$

By the Quadratic Formula: $a \approx 14$ or $a \approx 6$. Choosing the larger value of a, we have $a \approx 14$ and $b \approx 6$. The equation of an ellipse with an area of 264 is

$$\frac{x^2}{196} + \frac{y^2}{36} = 1.$$

—CONTINUED—

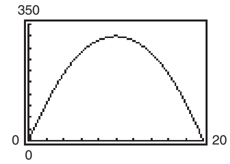
69. —CONTINUED—

(c)

a	8	9	10	11	12	13
A	301.6	311.0	314.2	311.0	301.6	285.9

The area is maximum when $a = 10$ and the ellipse is a circle.

(d)



The area is maximum (314.16) when $a = b = 10$ and the ellipse is a circle.

70. (a) Length of string $= 2a$

(b) By keeping the string taut, the sum of the distances from the two fixed points is constant (equal to the length of the string).

71. 80, 40, 20, 10, 5, . . .

Geometric, $r = \frac{1}{2}$

72. 66, 55, 44, 33, 22, . . .

Arithmetic sequence

73. $-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \ldots$

Arithmetic, $d = 1$

74. $\frac{1}{4}, \frac{1}{2}, 1, 2, 4$

Geometric sequence

75. $\displaystyle\sum_{n=0}^{6} (-3)^n = 1 - 3 + 9 - 27 + 81 - 243 + 729$

$= 547$

76. $\displaystyle\sum_{n=0}^{6} 3^n = \sum_{n=1}^{7} 3^{(n-1)} \implies a_1 = 1, r = 3$

$$S_7 = \frac{1(1 - 3^7)}{1 - 3} = 1093$$

77. $\displaystyle\sum_{n=0}^{10} 5\left(\frac{4}{3}\right)^n = 5\frac{\left(1 - \left(\frac{4}{3}\right)^{11}\right)}{1 - \frac{4}{3}}$

≈ 340.15

78. $\displaystyle\sum_{n=1}^{10} 4\left(\frac{3}{4}\right)^{n-1} \implies a_1 = 4, r = \frac{3}{4}$

$$S_{10} = \frac{4\left(1 - \left(\frac{3}{4}\right)^{10}\right)}{1 - \frac{3}{4}} \approx 15.10$$

Section 10.4 Hyperbolas

- A **hyperbola** is the set of all points (x, y) the difference of whose distances from two distinct fixed points **(foci)** is constant.
- The standard equation of a hyperbola with center (h, k) and transverse and conjugate axes of lengths $2a$ and $2b$ is:
  (a) $\dfrac{(x - h)^2}{a^2} - \dfrac{(y - k)^2}{b^2} = 1$ if the traverse axis is horizontal.
  (b) $\dfrac{(y - k)^2}{a^2} - \dfrac{(x - h)^2}{b^2} = 1$ if the traverse axis is vertical.
- $c^2 = a^2 + b^2$ where c is the distance from the center to a focus.
- The asymptotes of a hyperbola are:
  (a) $y = k \pm \dfrac{b}{a}(x - h)$ if the transverse axis is horizontal.
  (b) $y = k \pm \dfrac{a}{b}(x - h)$ if the transverse axis is vertical.
- The eccentricity of a hyperbola is $e = \dfrac{c}{a}$.
- To classify a nondegenerate conic from its general equation $Ax^2 + Cy^2 + Dx + Ey + F = 0$:
  (a) If $A = C$ $(A \neq 0, C \neq 0)$, then it is a circle.
  (b) If $AC = 0$ $(A = 0$ or $C = 0$, but not both), then it is a parabola.
  (c) If $AC > 0$, then it is an ellipse.

Vocabulary Check

1. hyperbola
2. branches
3. transverse axis; center
4. asymptotes
5. $Ax^2 + Cy^2 + Dx + Ey + F = 0$

1. $\dfrac{y^2}{9} - \dfrac{x^2}{25} = 1$

Center: $(0, 0)$

$a = 3, b = 5$

Vertical transverse axis

Matches graph (b).

2. $\dfrac{y^2}{25} - \dfrac{x^2}{9} = 1$

Center: $(0, 0)$

$a = 5, b = 3$

Vertical transverse axis

Matches graph (c).

3. $\dfrac{(x-1)^2}{16} - \dfrac{y^2}{4} = 1$

Center: $(1, 0)$

$a = 4, b = 2$

Horizontal transverse axis

Matches graph (a).

4. $\dfrac{(x+1)^2}{16} - \dfrac{(y-2)^2}{9} = 1$

Center: $(-1, 2)$

$a = 4, b = 3$

Horizontal transverse axis

Matches graph (d).

5. $x^2 - y^2 = 1$

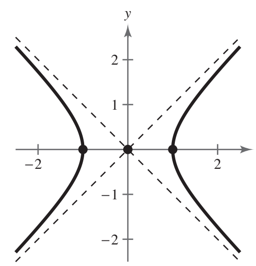
$a = 1, b = 1, c = \sqrt{2}$

Center: $(0, 0)$

Vertices: $(\pm 1, 0)$

Foci: $\left(\pm\sqrt{2}, 0\right)$

Asymptotes: $y = \pm x$



6. $\dfrac{x^2}{9} - \dfrac{y^2}{25} = 1$

$a = 3, b = 5$

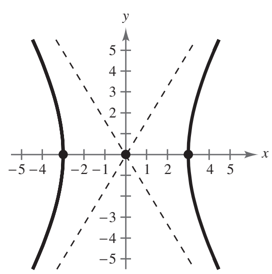
$c = \sqrt{3^2 + 5^2} = \sqrt{34}$

Center: $(0, 0)$

Vertices: $(\pm 3, 0)$

Foci: $\left(\pm\sqrt{34}, 0\right)$

Asymptotes: $y = \pm\dfrac{5}{3}x$



7. $\dfrac{y^2}{25} - \dfrac{x^2}{81} = 1$

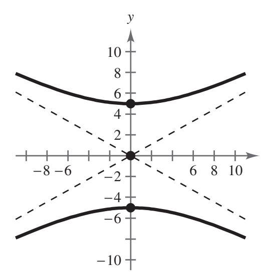
$a = 5, b = 9, c = \sqrt{106}$

Center: $(0, 0)$

Vertices: $(0, \pm 5)$

Foci: $\left(0, \pm\sqrt{106}\right)$

Asymptotes: $y = \pm\dfrac{5}{9}x$



8. $\dfrac{x^2}{36} - \dfrac{y^2}{4} = 1$

$a = 6, b = 2,$

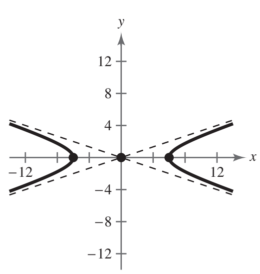
$c = \sqrt{36 + 4} = 2\sqrt{10}$

Center: $(0, 0)$

Vertices: $(\pm 6, 0)$

Foci: $\left(\pm 2\sqrt{10}, 0\right)$

Asymptotes: $y = \pm\dfrac{1}{3}x$



9. $\dfrac{(x-1)^2}{4} - \dfrac{(y+2)^2}{1} = 1$

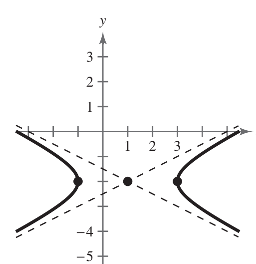
$a = 2, b = 1, c = \sqrt{5}$

Center: $(1, -2)$

Vertices: $(-1, -2), (3, -2)$

Foci: $\left(1 \pm \sqrt{5}, -2\right)$

Asymptotes: $y = -2 \pm \dfrac{1}{2}(x - 1)$



10. $\dfrac{(x+3)^2}{144} - \dfrac{(y-2)^2}{25} = 1$

$a = 12, b = 5$

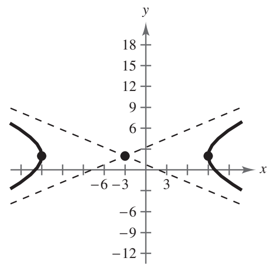
$c = \sqrt{144 + 25} = 13$

Center: $(-3, 2)$

Vertices: $(9, 2), (-15, 2)$

Foci: $(10, 2), (-16, 2)$

Asymptotes: $y = 2 \pm \dfrac{5}{12}(x + 3)$



11. $\dfrac{(y+6)^2}{1/9} - \dfrac{(x-2)^2}{1/4} = 1$

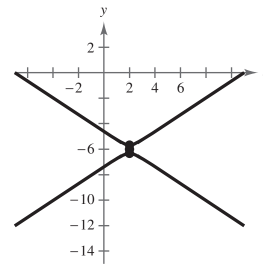
$a = \dfrac{1}{3}, b = \dfrac{1}{2}, c = \dfrac{\sqrt{13}}{6}$

Center: $(2, -6)$

Vertices: $\left(2, -\dfrac{17}{3}\right), \left(2, -\dfrac{19}{3}\right)$

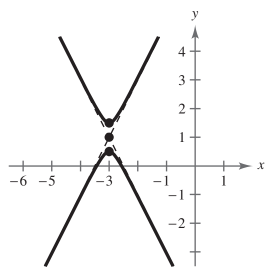
Foci: $\left(2, -6 \pm \dfrac{\sqrt{13}}{6}\right)$

Asymptotes: $y = -6 \pm \dfrac{2}{3}(x - 2)$



12. $\dfrac{(y-1)^2}{1/4} - \dfrac{(x+3)^2}{1/16} = 1$

$a = \dfrac{1}{2}, b = \dfrac{1}{4}$



$c = \sqrt{\dfrac{1}{4} + \dfrac{1}{16}} = \dfrac{\sqrt{5}}{4}$

Center: $(-3, 1)$

13.

$$9x^2 - y^2 - 36x - 6y + 18 = 0$$

$$9(x^2 - 4x + 4) - (y^2 + 6y + 9) = -18 + 36 - 9$$

$$9(x-2)^2 - (y+3)^2 = 9$$

$$\frac{(x-2)^2}{1} - \frac{(y+3)^2}{9} = 1$$

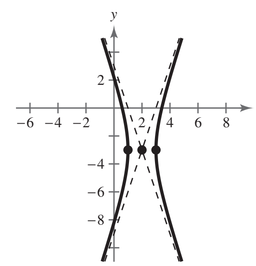
$a = 1, b = 3, c = \sqrt{10}$

Center: $(2, -3)$

Vertices: $(1, -3), (3, -3)$

Foci: $\left(2 \pm \sqrt{10}, -3\right)$

Asymptotes:
$y = -3 \pm 3(x - 2)$



14. $x^2 - 9y^2 + 36y - 72 = 0$

$$x^2 - 9(y^2 - 4y + 4) = 72 - 36$$

$$x^2 - 9(y-2)^2 = 36$$

$$\frac{x^2}{36} - \frac{(y-2)^2}{4} = 1$$

$a = 6, b = 2,$

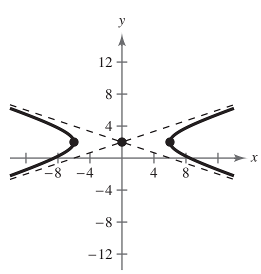
$c = \sqrt{36 + 4} = 2\sqrt{10}$

Center: $(0, 2)$

Vertices: $(\pm 6, 2)$

Foci: $\left(\pm 2\sqrt{10}, 2\right)$

Asymptotes: $y = 2 \pm \dfrac{1}{3}x$



15.

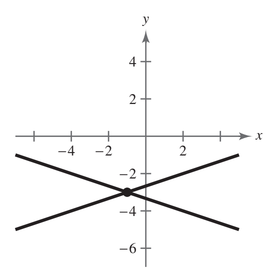
$$x^2 - 9y^2 + 2x - 54y - 80 = 0$$

$$(x^2 + 2x + 1) - 9(y^2 + 6y + 9) = 80 + 1 - 81$$

$$(x+1)^2 - 9(y+3)^2 = 0$$

$$y + 3 = \pm\tfrac{1}{3}(x + 1)$$

Degenerate hyperbola is two lines intersecting at $(-1, -3)$.



16.

$$16y^2 - x^2 + 2x + 64y + 63 = 0$$

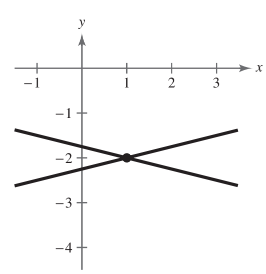
$$16(y^2 + 4y + 4) - (x^2 - 2x + 1) = -63 + 64 - 1$$

$$16(y+2)^2 - (x-1)^2 = 0$$

$$y + 2 = \pm\tfrac{1}{4}(x - 1)$$

Degenerate hyperbola

The graph is two lines intersecting at $(1, -2)$.



17. $2x^2 - 3y^2 = 6$

$$\frac{x^2}{3} - \frac{y^2}{2} = 1$$

$a = \sqrt{3}, b = \sqrt{2}, c = \sqrt{5}$

Center: $(0, 0)$

Vertices: $(\pm\sqrt{3}, 0)$

Foci: $(\pm\sqrt{5}, 0)$

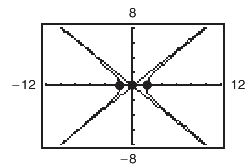
Asymptotes: $y = \pm\sqrt{\frac{2}{3}}x = \pm\frac{\sqrt{6}}{3}x$

To use a graphing calculator, solve for y first.

$$y^2 = \frac{2x^2 - 6}{3}$$

$$\left.\begin{aligned} y_1 &= \sqrt{\frac{2x^2-6}{3}} \\ y_2 &= -\sqrt{\frac{2x^2-6}{3}} \end{aligned}\right\} \text{Hyperbola}$$

$$\left.\begin{aligned} y_3 &= \frac{\sqrt{6}}{3}x \\ y_4 &= -\frac{\sqrt{6}}{3}x \end{aligned}\right\} \text{Asymptotes}$$



18. $6y^2 - 3x^2 = 18$

$$\frac{y^2}{3} - \frac{x^2}{6} = 1$$

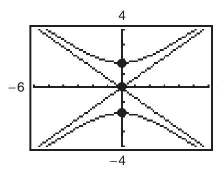
$a = \sqrt{3}, b = \sqrt{6}, c = \sqrt{3 + 6} = 3$

Center: $(0, 0)$

Vertices: $(0, \pm\sqrt{3})$

Foci: $(0, \pm 3)$

Asymptotes: $y = \pm\frac{\sqrt{2}}{2}x$



19.

$$\begin{aligned} 9y^2 - x^2 + 2x + 54y + 62 &= 0 \\ 9(y^2 + 6y + 9) - (x^2 - 2x + 1) &= -62 - 1 + 81 \\ 9(y + 3)^2 - (x - 1)^2 &= 18 \\ \frac{(y+3)^2}{2} - \frac{(x-1)^2}{18} &= 1 \end{aligned}$$

$a = \sqrt{2}, b = 3\sqrt{2}, c = 2\sqrt{5}$

Center: $(1, -3)$

Vertices: $(1, -3 \pm \sqrt{2})$

Foci: $(1, -3 \pm 2\sqrt{5})$

Asymptotes: $y = -3 \pm \frac{1}{3}(x - 1)$

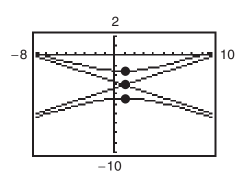
To use a graphing calculator, solve for y first.

$$9(y + 3)^2 = 18 + (x - 1)^2$$

$$y = -3 \pm \sqrt{\frac{18 + (x-1)^2}{9}}$$

$$\left.\begin{aligned} y_1 &= -3 + \frac{1}{3}\sqrt{18 + (x-1)^2} \\ y_2 &= -3 - \frac{1}{3}\sqrt{18 + (x-1)^2} \end{aligned}\right\} \text{Hyperbola}$$

$$\left.\begin{aligned} y_3 &= -3 + \frac{1}{3}(x - 1) \\ y_4 &= -3 - \frac{1}{3}(x - 1) \end{aligned}\right\} \text{Asymptotes}$$



20.
$$9x^2 - y^2 + 54x + 10y + 55 = 0$$
$$9(x^2 + 6x + 9) - (y^2 - 10y + 25) = -55 + 81 - 25$$
$$\frac{(x+3)^2}{1/9} - \frac{(y-5)^2}{1} = 1$$

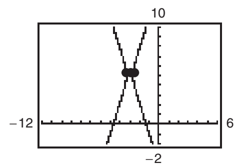
$a = \frac{1}{3}, b = 1, c = \frac{\sqrt{10}}{3}$

Center: $(-3, 5)$

Vertices: $\left(-3 \pm \frac{1}{3}, 5\right) \Rightarrow \left(-\frac{10}{3}, 5\right), \left(-\frac{8}{3}, 5\right)$

Foci: $\left(-3 \pm \frac{\sqrt{10}}{3}, 5\right)$

Asymptotes:
$y = 5 \pm 3(x + 3)$



21. Vertices: $(0, \pm 2) \Rightarrow a = 2$

Foci: $(0, \pm 4) \Rightarrow c = 4$

$b^2 = c^2 - a^2 = 16 - 4 = 12$

Center: $(0, 0) = (h, k)$

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

$$\frac{y^2}{4} - \frac{x^2}{12} = 1$$

22. Vertices: $(\pm 4, 0) \Rightarrow a = 4$

Foci: $(\pm 6, 0) \Rightarrow c = 6$

$b^2 = c^2 - a^2 = 36 - 16 = 20 \Rightarrow b = 2\sqrt{5}$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{16} - \frac{y^2}{20} = 1$$

23. Vertices: $(\pm 1, 0) \Rightarrow a = 1$

Asymptotes: $y = \pm 5x \Rightarrow \frac{b}{a} = 5, b = 5$

Center: $(0, 0) = (h, k)$

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

$$\frac{x^2}{1} - \frac{y^2}{25} = 1$$

24. Vertices: $(0, \pm 3) \Rightarrow a = 3$

Asymptotes: $y = \pm 3x \Rightarrow \frac{a}{b} = 3, b = 1$

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

$$\frac{y^2}{9} - \frac{x^2}{1} = 1$$

25. Foci: $(0, \pm 8) \Rightarrow c = 8$

Asymptotes: $y = \pm 4x \Rightarrow \frac{a}{b} = 4 \Rightarrow a = 4b$

Center: $(0, 0) = (h, k)$

$c^2 = a^2 + b^2 \Rightarrow 64 = 16b^2 + b^2$

$$\frac{64}{17} = b^2 \Rightarrow a^2 = \frac{1024}{17}$$

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

$$\frac{y^2}{1024/17} - \frac{x^2}{64/17} = 1$$

$$\frac{17y^2}{1024} - \frac{17x^2}{64} = 1$$

26. Foci: $(\pm 10, 0) \Rightarrow c = 10$

Asymptotes: $y = \pm\frac{3}{4}x \Rightarrow \frac{b}{a} = \frac{3m}{4m}$

$c^2 = a^2 + b^2 \Rightarrow 100 = (3m)^2 + (4m)^2$

$100 = 25m^2$

$2 = m$

$a = 4(2) = 8$

$b = 3(2) = 6$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{64} - \frac{y^2}{36} = 1$$

27. Vertices: $(2, 0), (6, 0) \implies a = 2$

Foci: $(0, 0), (8, 0) \implies c = 4$

$b^2 = c^2 - a^2 = 16 - 4 = 12$

Center: $(4, 0) = (h, k)$

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

$$\frac{(x-4)^2}{4} - \frac{y^2}{12} = 1$$

28. Vertices: $(2, 3), (2, -3) \implies a = 3$

Center: $(2, 0)$

Foci: $(2, 6), (2, -6) \implies c = 6$

$b^2 = c^2 - a^2 = 36 - 9 = 27$

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

$$\frac{y^2}{9} - \frac{(x-2)^2}{27} = 1$$

29. Vertices: $(4, 1), (4, 9) \implies a = 4$

Foci: $(4, 0), (4, 10) \implies c = 5$

$b^2 = c^2 - a^2 = 25 - 16 = 9$

Center: $(4, 5) = (h, k)$

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

$$\frac{(y-5)^2}{16} - \frac{(x-4)^2}{9} = 1$$

30. Vertices: $(-2, 1), (2, 1) \implies a = 2$

Center: $(0, 1)$

Foci: $(-3, 1), (3, 1) \implies c = 3$

$b^2 = c^2 - a^2 = 9 - 4 = 5$

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

$$\frac{x^2}{4} - \frac{(y-1)^2}{5} = 1$$

31. Vertices: $(2, 3), (2, -3) \implies a = 3$

Passes through the point: $(0, 5)$

Center: $(2, 0) = (h, k)$

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

$$\frac{y^2}{9} - \frac{(x-2)^2}{b^2} = 1 \implies$$

$$\frac{(x-2)^2}{b^2} = \frac{y^2}{9} - 1 = \frac{y^2-9}{9} \implies$$

$$b^2 = \frac{9(x-2)^2}{y^2-9} = \frac{9(-2)^2}{25-9} = \frac{36}{16} = \frac{9}{4}$$

$$\frac{y^2}{9} - \frac{(x-2)^2}{9/4} = 1$$

$$\frac{y^2}{9} - \frac{4(x-2)^2}{9} = 1$$

32. Vertices: $(-2, 1), (2, 1) \implies a = 2$

Center: $(0, 1)$

Point on curve: $(5, 4)$

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

$$\frac{x^2}{4} - \frac{(y-1)^2}{b^2} = 1$$

$$\frac{25}{4} - \frac{9}{b^2} = 1$$

$$b^2 = \frac{12}{7}$$

$$\frac{x^2}{4} - \frac{(y-1)^2}{12/7} = 1$$

$$\frac{x^2}{4} - \frac{7(y-1)^2}{12} = 1$$

33. Vertices: $(0, 4), (0, 0) \implies a = 2$

Passes through the point $\left(\sqrt{5}, -1\right)$

Center: $(0, 2) = (h, k)$

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

$$\frac{(y-2)^2}{4} - \frac{x^2}{b^2} = 1 \implies \frac{x^2}{b^2} = \frac{(y-2)^2}{4} - 1 = \frac{(y-2)^2-4}{4}$$

$$\implies b^2 = \frac{4x^2}{(y-2)^2-4} = \frac{4\left(\sqrt{5}\right)^2}{(-1-2)^2-4} = \frac{20}{5} = 4$$

$$\frac{(y-2)^2}{4} - \frac{x^2}{4} = 1$$

34. Vertices: $(1, \pm 2) \implies a = 2$

Center: $(1, 0)$

Point on curve: $\left(0, \sqrt{5}\right)$

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

$$\frac{y^2}{4} - \frac{(x-1)^2}{b^2} = 1$$

$$\frac{5}{4} - \frac{1}{b^2} = 1$$

$$b^2 = 4$$

$$\frac{y^2}{4} - \frac{(x-1)^2}{4} = 1$$

35. Vertices: $(1, 2), (3, 2) \implies a = 1$

Asymptotes: $y = x, y = 4 - x$

$$\frac{b}{a} = 1 \implies \frac{b}{1} = 1 \implies b = 1$$

Center: $(2, 2) = (h, k)$

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

$$\frac{(x-2)^2}{1} - \frac{(y-2)^2}{1} = 1$$

36. Vertices: $(3, 0), (3, 6) \implies a = 3$

Center: $(3, 3)$

Asymptotes: $y = 6 - x, y = x$

$$\frac{a}{b} = 1 \implies b = 3$$

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

$$\frac{(y-3)^2}{9} - \frac{(x-3)^2}{9} = 1$$

37. Vertices: $(0, 2), (6, 2) \implies a = 3$

Asymptotes: $y = \frac{2}{3}x, y = 4 - \frac{2}{3}x$

$$\frac{b}{a} = \frac{2}{3} \implies b = 2$$

Center: $(3, 2) = (h, k)$

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

$$\frac{(x-3)^2}{9} - \frac{(y-2)^2}{4} = 1$$

38. Vertices: $(3, 0), (3, 4) \implies a = 2$

Asymptotes: $y = \frac{2}{3}x, y = 4 - \frac{2}{3}x$

$$\frac{a}{b} = \frac{2}{3} \implies b = 3$$

Center: $(3, 2) = (h, k)$

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

$$\frac{(y-2)^2}{4} - \frac{(x-3)^2}{9} = 1$$

39. (a) Vertices: $(\pm 1, 0) \implies a = 1$

Horizontal transverse axis

Center: $(0, 0)$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Point on the graph: $(2, 13)$

$$\frac{2^2}{1^2} - \frac{13^2}{b^2} = 1$$

$$4 - \frac{169}{b^2} = 1$$

$$3b^2 = 169$$

$$b^2 = \frac{169}{3} \approx 56.33$$

Thus we have $\frac{x^2}{1} - \frac{y^2}{56.33} = 1.$

(b) When $y = 5$: $x^2 = 1 + \frac{5^2}{56.33}$

$$x = \sqrt{1 + \frac{25}{56.33}} \approx 1.2016$$

Width: $2x \approx 2.403$ feet

40. $2c = 4 \text{ mi} = 21{,}120 \text{ ft}$

$c = 10{,}560 \text{ ft}$

$(1100 \text{ ft/s})(18 \text{ s}) = 19{,}800 \text{ ft}$

The lightening occurred 19,800 feet further from B than from A:

$d_2 - d_1 = 2a = 19{,}800 \text{ ft}$

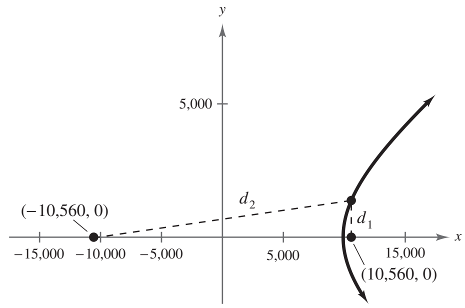
$a = 9900 \text{ ft}$

$b^2 = c^2 - a^2 = (10{,}560)^2 - (9900)^2$

$b^2 = 13{,}503{,}600$

$$\frac{x^2}{(9900)^2} - \frac{y^2}{13{,}503{,}600} = 1$$

$$\frac{x^2}{98{,}010{,}000} - \frac{y^2}{13{,}503{,}600} = 1$$



41. Since listening station C heard the explosion 4 seconds after listening station A, and since listening station B heard the explosion one second after listening station A, and sound travels 1100 feet per second, the explosion is located in Quadrant IV on the line $x = 3300$. The locus of all points 4400 feet closer to A than C is one branch of the hyperbola.

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where $c = 3300$ feet and $a = \frac{4400}{2} = 2200$ feet, $b^2 = c^2 - a^2 = 6{,}050{,}000$.

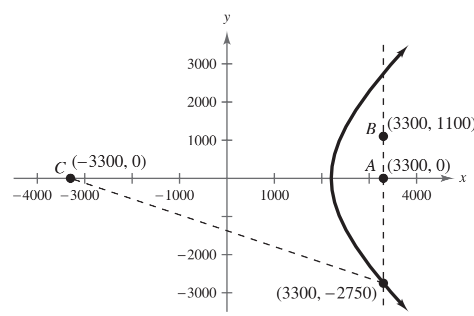
When $x = 3300$ we have $\frac{3300^2}{2200^2} - \frac{y^2}{6{,}050{,}000} = 1$.

Solving for y: $y^2 = 6{,}050{,}000\left(\frac{3300^2}{2200^2} - 1\right)$

$= 7{,}562{,}500$

$y = \pm 2750$

Since the explosion is in Quadrant IV, its coordinates are $(3300, -2750)$.



42. (a) Foci: $(\pm 150, 0) \implies c = 150$

Center: $(0, 0) = (h, k)$

$\frac{d_2}{186{,}000} - \frac{d_1}{186{,}000} = 0.001 \implies 2a = 186,\ a = 93$

$b^2 = c^2 - a^2 = 150^2 - 93^2 = 13{,}851$

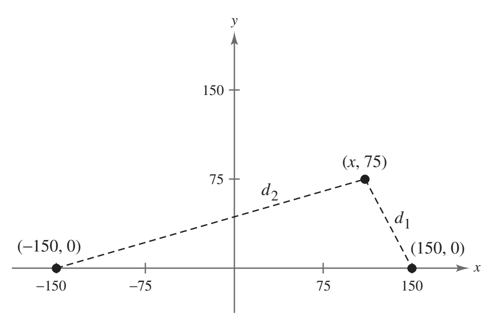
$$\frac{x^2}{93^2} - \frac{y^2}{13{,}851} = 1$$

$$x^2 = 93^2\left(1 + \frac{75^2}{13{,}851}\right) \approx 12{,}161$$

$x \approx 110.3$ miles

(b) $c - a = 150 - 93 = 57$ miles

(c) $\frac{270}{186{,}000} - \frac{30}{186{,}000} \approx 0.00129$ second



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42. —CONTINUED—

(d) $\dfrac{d_2}{186{,}000} - \dfrac{d_1}{186{,}000} = 0.00129$

$2a \approx 239.94$

$a \approx 119.97$

$b^2 = c^2 - a^2 = 150^2 - 119.97^2 = 8107.1991$

$$\frac{x^2}{119.97^2} - \frac{y^2}{8107.1991} = 1$$

$$x^2 = 119.97^2\left(1 + \frac{60^2}{8107.1991}\right)$$

$$x \approx 144.2 \text{ miles}$$

Position: $(144.2, 60)$

43. Center: $(0, 0) = (h, k)$

Focus: $(24, 0) \implies c = 24$

Solution point: $(24, 24)$

$24^2 = a^2 + b^2 \implies b^2 = 24^2 - a^2$

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

$$\frac{x^2}{a^2} - \frac{y^2}{24^2 - a^2} = 1 \implies \frac{24^2}{a^2} - \frac{24^2}{24^2 - a^2} = 1$$

Solving yields $a = 12\sqrt{2\left(3 - \sqrt{5}\right)}$ OR $12\left(\sqrt{5} - 1\right) \approx 14.83$ and $b^2 \approx 355.9876$.

Thus, we have $\dfrac{x^2}{220.0124} - \dfrac{y^2}{355.9876} = 1.$

The right vertex is at $(a, 0) \approx (14.83, 0)$.

44. (a) $x^2 + y^2 - 200x - 52{,}500 = 0$

$Ax^2 + Cy^2 + Dx + Ey + F = 0$

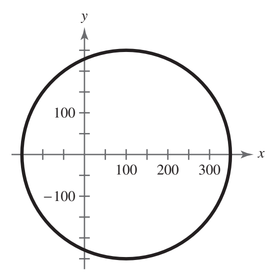
$A = 1, C = 1, D = -200, E = 0, F = -52{,}500$

$A = C$: circle

(b) $(x^2 - 200x + 10{,}000) + (y^2) = 52{,}500 + 10{,}000$

$$(x - 100)^2 + y^2 = 62{,}500$$

$$\frac{(x-100)^2}{62{,}500} + \frac{y^2}{62{,}500} = 1$$



(c) $d = \sqrt{(-100 - 0)^2 + (150 - 0)^2} = 50\sqrt{13}$

$d \approx 180.28$ meters

45. $x^2 + y^2 - 6x + 4y + 9 = 0$

$A = 1, C = 1$

$A = C \implies$ Circle

46. $x^2 + 4y^2 - 6x + 16y + 21 = 0$

$A = 1, C = 4$

$AC = (1)(4) > 0$ and $A \neq C \implies$ Ellipse

47. $4x^2 - y^2 - 4x - 3 = 0$

$A = 4, C = -1$

$AC = (4)(-1) = -4 < 0 \implies$ Hyperbola

48. $y^2 - 6y - 4x + 21 = 0$

$A = 0, C = 1$

$AC = (0)(1) = 0 \implies$ Parabola

49. $y^2 - 4x^2 + 4x - 2y - 4 = 0$

$A = -4, C = 1$

$AC = (-4)(1) = -4 < 0 \implies$ Hyperbola

50. $x^2 + y^2 - 4x + 6y - 3 = 0$

$A = 1, C = 1$

$A = C \implies$ Circle

51. $x^2 - 4x - 8y + 2 = 0$

$A = 1, C = 0$

$AC = (1)(0) = 0 \implies$ Parabola

52. $4x^2 + y^2 - 8x + 3 = 0$

$A = 4, C = 1$

$AC = 4 > 0$ and $A \neq C \implies$ Ellipse

53. $4x^2 + 3y^2 + 8x - 24y + 51 = 0$

$A = 4, C = 3$

$AC = 4(3) = 12 > 0$ and $A \neq C \implies$ Ellipse

54. $4y^2 - 2x^2 - 4y - 8x - 15 = 0$

$AC = (-2)(4) < 0 \implies$ Hyperbola

55. $25x^2 - 10x - 200y - 119 = 0$

$A = 25, C = 0$

$AC = 25(0) = 0 \implies$ Parabola

56. $4y^2 + 4x^2 - 24x + 35 = 0$

$A = C = 4 \implies$ Circle

57. $4x^2 + 16y^2 - 4x - 32y + 1 = 0$

$A = 4, C = 16$

$AC = (4)(16) = 64 > 0$ and $A \neq C \implies$ Ellipse

58. $2y^2 + 2x + 2y + 1 = 0$

$A = 0, C = 2$

$AC = 0$, but $C \neq 0 \implies$ Parabola

59. $100x^2 + 100y^2 - 100x + 400y + 409 = 0$

$A = 100, C = 100$

$A = C \implies$ Circle

60. $4x^2 - y^2 + 4x + 2y - 1 = 0$

$A = 4, C = -1$

$AC = (4)(-1) = -4 < 0 \implies$ Hyperbola

61. True. For a hyperbola, $c^2 = a^2 + b^2$ or

$$e^2 = \frac{c^2}{a^2} = 1 + \frac{b^2}{a^2}.$$

The larger the ratio of b to a, the larger the eccentricity $e = c/a$ of the hyperbola.

62. False. For the trivial solution of two intersecting lines to occur, the standard form of the equation of the hyperbola would be equal to zero.

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 0 \quad \text{or} \quad \frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 0$$

63. Let (x, y) be such that the difference of the distances from $(c, 0)$ and $(-c, 0)$ is $2a$ (again only deriving one of the forms).

$$2a = \left|\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2}\right|$$

$$\begin{aligned}
2a + \sqrt{(x-c)^2 + y^2} &= \sqrt{(x+c)^2 + y^2} \\
4a^2 + 4a\sqrt{(x-c)^2 + y^2} + (x-c)^2 + y^2 &= (x+c)^2 + y^2 \\
4a\sqrt{(x-c)^2 + y^2} &= 4cx - 4a^2 \\
a\sqrt{(x-c)^2 + y^2} &= cx - a^2 \\
a^2(x^2 - 2cx + c^2 + y^2) &= c^2x^2 - 2a^2cx + a^4 \\
a^2(c^2 - a^2) &= (c^2 - a^2)x^2 - a^2y^2
\end{aligned}$$

Let $b^2 = c^2 - a^2$. Then $a^2b^2 = b^2x^2 - a^2y^2 \implies 1 = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}.$

64. The extended diagonals of the central rectangle are the asymptotes of the hyperbola.

65.

$$9x^2 - 54x - 4y^2 + 8y + 41 = 0$$

$$9(x^2 - 6x + 9) - 4(y^2 - 2y + 1) = -41 + 81 - 4$$

$$9(x - 3)^2 - 4(y - 1)^2 = 36$$

$$\frac{(x - 3)^2}{4} - \frac{(y - 1)^2}{9} = 1$$

$$\frac{(y - 1)^2}{9} = \frac{(x - 3)^2}{4} - 1$$

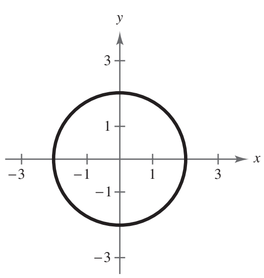
$$(y - 1)^2 = 9\left[\frac{(x - 3)^2}{4} - 1\right]$$

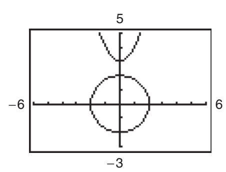
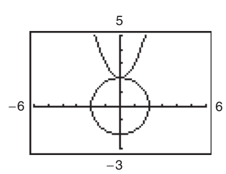
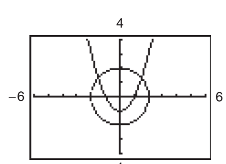
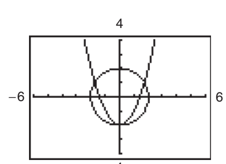
The bottom half of the hyperbola is:

$$y - 1 = -\sqrt{9\left[\frac{(x - 3)^2}{4} - 1\right]}$$

$$y = 1 - 3\sqrt{\frac{(x - 3)^2}{4} - 1}$$

66.



Value of C	Possible number of points of intersection
$C > 2$	
$C = 2$	
$-2 < C < 2$	
$C = -2$	

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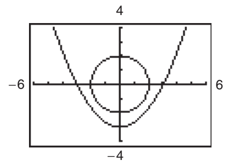
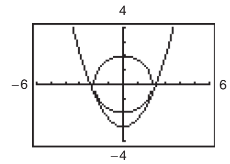
66. —CONTINUED—

$C < -2$



or



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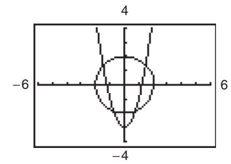


For $C \le -2$, we need to analyze the two curves to determine the number of points of intersection.

$C = -2$:

$$x^2 + y^2 = 4 \quad \text{and} \quad y = x^2 - 2$$
$$x^2 = y + 2$$

Substitute: $(y + 2) + y^2 = 4$

$$y^2 + y - 2 = 0$$
$$(y + 2)(y - 1) = 0$$
$$y = -2, 1$$

$$\begin{array}{ll} x^2 = y + 2 & x^2 = y + 2 \\ x^2 = -2 + 2 & x^2 = 1 + 2 \\ x^2 = 0 & x^2 = 3 \\ x = 0 & x = \pm\sqrt{3} \\ (0, -2) & \left(-\sqrt{3}, 1\right), \left(\sqrt{3}, 1\right) \end{array}$$

There are three points of intersection when $C = -2$.

$C < -2$:

$$x^2 + y^2 = 4 \quad \text{and} \quad y = x^2 + C$$
$$x^2 = y - C$$

Substitute: $(y - C) + y^2 = 4$

$$y^2 + y - 4 - C = 0$$
$$y = \frac{-1 \pm \sqrt{(1)^2 - (4)(1)(-C - 4)}}{2}$$
$$y = \frac{-1 \pm \sqrt{1 + 4(C + 4)}}{2}$$

If $1 + 4(C + 4) < 0$, there are no real solutions (no points of intersection):

$$1 + 4C + 16 < 0$$
$$4C < -17$$
$$C < \frac{-17}{4}, \text{ no points of intersection}$$

If $1 + 4(C + 4) = 0$, there is one real solution (two points of intersection):

$$1 + 4C + 16 = 0$$
$$4C = -17$$
$$C = \frac{-17}{4}, \text{ two points of intersection}$$

—CONTINUED—

66. —CONTINUED—

If $1 + 4(C + 4) > 0$, there are two real solutions (four points of intersection):

$$1 + 4C + 16 > 0$$
$$4C > -17$$
$$C > \frac{-17}{4}, \text{ (but } C < -2\text{), four points of intersection}$$

Summary:

a. no points of intersection: $C > 2$ or $C < \frac{-17}{4}$

b. one point of intersection: $C = 2$

c. two points of intersection: $-2 < C < 2$ or $C = \frac{-17}{4}$

d. three points of intersection: $C = -2$

e. four points of intersection: $\frac{-17}{4} < C < -2$

67. $x^3 - 16x = x(x^2 - 16) = x(x + 4)(x - 4)$

68. $x^2 + 14x + 49 = x^2 + 2(7)x + 7^2 = (x + 7)^2$

69. $2x^3 - 24x^2 + 72x = 2x(x^2 - 12x + 36) = 2x(x - 6)^2$

70.
$$\begin{aligned} 6x^3 - 11x^2 - 10x &= x(6x^2 - 11x - 10) \\ &= x(6x^2 - 15x + 4x - 10) \\ &= x[3x(2x - 5) + 2(2x - 5)] \\ &= x(3x + 2)(2x - 5) \end{aligned}$$

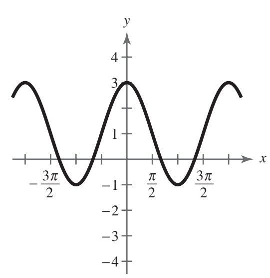
71.
$$\begin{aligned} 16x^3 + 54 &= 2(8x^3 + 27) \\ &= 2[(2x)^3 + (3)^3] \\ &= 2(2x + 3)(4x^2 - 6x + 9) \end{aligned}$$

72.
$$\begin{aligned} 4 - x + 4x^2 - x^3 &= (4 + 4x^2) - (x + x^3) \\ &= 4(1 + x^2) - x(1 + x^2) \\ &= (4 - x)(1 + x^2) \end{aligned}$$

73. $y = 2 \cos x + 1$

Amplitude: 2

Period: 2π

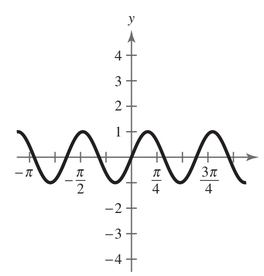


74. $y = \sin \pi x$

Period: $\frac{2\pi}{\pi} = 2$

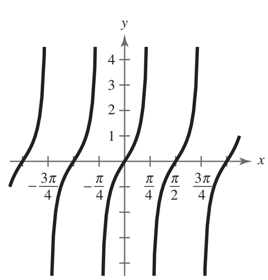
Amplitude: 1

Key points: $(0, 0)$, $\left(\frac{1}{2}, 1\right)$, $(1, 0)$, $\left(\frac{3}{2}, -1\right)$, $(2, 0)$



75. $y = \tan 2x$

Period: $\frac{\pi}{2}$

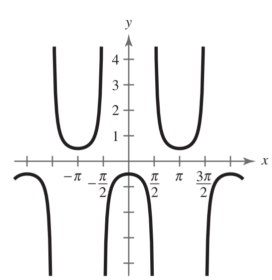


76. $y = -\frac{1}{2} \sec x$

Graph $y = -\frac{1}{2} \cos x$ first.

Period: 2π

One cycle: 0 to 2π



Section 10.5 Rotation of Conics

- The general second-degree equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ can be rewritten as $A'(x')^2 + C'(y')^2 + D'x' + E'y' + F' = 0$ by rotating the coordinate axes through the angle θ, where $\cot 2\theta = (A - C)/B$ and the following quantities are invariant under rotation:
  1. $F = F'$
  2. $A + C = A' + C'$
  3. $B^2 - 4AC = (B')^2 - 4A'C'$
- $x = x'\cos\theta - y'\sin\theta$
  $y = x'\sin\theta + y'\cos\theta$
- The graph of the nondegenerate equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ is:
  (a) An ellipse or circle if $B^2 - 4AC < 0$.
  (b) A parabola if $B^2 - 4AC = 0$.
  (c) A hyperbola if $B^2 - 4AC > 0$.

Vocabulary Check

1. rotation of axes
2. $A'(x')^2 + C'(y')^2 + D'x' + E'y' + F' = 0$
3. invariant under rotation
4. discriminant

1. $\theta = 90°$; Point: $(0, 3)$

$x = x'\cos\theta - y'\sin\theta$ $\qquad$ $y = x'\sin\theta + y'\cos\theta$

$0 = x'\cos 90° - y'\sin 90°$ $\qquad$ $3 = x'\sin 90° - y'\cos 90°$

$0 = y'$ $\qquad$ $3 = x'$

So, $(x', y') = (3, 0)$.

2. $\theta = 45°$; Point: $(3, 3)$

$$\begin{matrix} x = x'\cos\theta - y'\sin\theta \\ y = x'\sin\theta + y'\cos\theta \end{matrix} \implies \begin{cases} 3 = x'\cos 45° - y'\sin 45° \\ 3 = x'\sin 45° + y'\cos 45° \end{cases}$$

Solving the system yields $(x', y') = \left(3\sqrt{2}, 0\right)$.

3. $\theta = 30°$; Point: $(1, 3)$

$$\begin{matrix} x = x'\cos\theta - y'\sin\theta \\ y = x'\sin\theta + y'\cos\theta \end{matrix} \implies \begin{cases} 1 = x'\cos 30° - y'\sin 30° \\ 3 = x'\sin 30° + y'\cos 30° \end{cases}$$

Solving the system yields $(x', y') = \left(\dfrac{3 + \sqrt{3}}{2}, \dfrac{3\sqrt{3} - 1}{2}\right)$.

4. $\theta = 60°$; Point: $(3, 1)$

$$\begin{matrix} x = x'\cos\theta - y'\sin\theta \\ y = x'\sin\theta + y'\cos\theta \end{matrix} \implies \begin{cases} 3 = x'\cos 60° - y'\sin 60° \\ 1 = x'\sin 60° + y'\cos 60° \end{cases}$$

Solving this system yields $(x', y') = \left(\dfrac{3 + \sqrt{3}}{2}, \dfrac{1 - 3\sqrt{3}}{2}\right)$.

5. $\theta = 45°$; Point $(2, 1)$

$$\begin{matrix} x = x'\cos\theta - y'\sin\theta \\ y = x'\sin\theta + y'\cos\theta \end{matrix} \implies \begin{cases} 2 = x'\cos 45° - y'\sin 45° \\ 1 = x'\sin 45° + y'\cos 45° \end{cases}$$

Solving the system yields $(x', y') = \left(\dfrac{3\sqrt{2}}{2}, -\dfrac{\sqrt{2}}{2}\right)$.

6. $\theta = 30°$; Point: $(2, 4)$

$$\begin{matrix} x = x'\cos\theta - y'\sin\theta \\ y = x'\sin\theta + y'\cos\theta \end{matrix} \implies \begin{cases} 2 = x'\cos 30° - y'\sin 30° \\ 4 = x'\sin 30° + y'\cos 30° \end{cases}$$

Solving this system yields $(x', y') = \left(\sqrt{3} + 2, 2\sqrt{3} - 1\right)$.

7. $xy + 1 = 0, A = 0, B = 1, C = 0$

$$\cot 2\theta = \frac{A - C}{B} = 0 \implies 2\theta = \frac{\pi}{2} \implies \theta = \frac{\pi}{4}$$

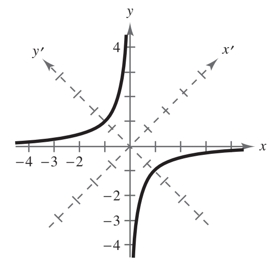
$$x = x'\cos\frac{\pi}{4} - y'\sin\frac{\pi}{4} = x'\left(\frac{\sqrt{2}}{2}\right) - y'\left(\frac{\sqrt{2}}{2}\right) = \frac{x' - y'}{\sqrt{2}}$$

$$y = x'\sin\frac{\pi}{4} + y'\cos\frac{\pi}{4} = x'\left(\frac{\sqrt{2}}{2}\right) + y'\left(\frac{\sqrt{2}}{2}\right) = \frac{x' + y'}{\sqrt{2}}$$

$$xy + 1 = 0$$

$$\left(\frac{x' - y'}{\sqrt{2}}\right)\left(\frac{x' + y'}{\sqrt{2}}\right) + 1 = 0$$

$$\frac{(y')^2}{2} - \frac{(x')^2}{2} = 1$$



8. $xy - 2 = 0, A = 0, B = 1, C = 0$

$$\cot 2\theta = \frac{A - C}{B} = 0 \implies 2\theta = \frac{\pi}{2} \implies \theta = \frac{\pi}{4}$$

$$x = x'\cos\frac{\pi}{4} - y'\sin\frac{\pi}{4} = \frac{x' - y'}{\sqrt{2}}$$

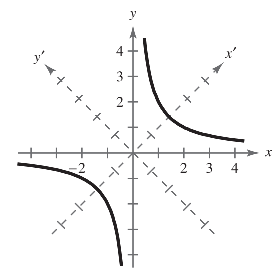
$$y = x'\sin\frac{\pi}{4} + y'\cos\frac{\pi}{4} = \frac{x' + y'}{\sqrt{2}}$$

$$xy - 2 = 0$$

$$\left(\frac{x' - y'}{\sqrt{2}}\right)\left(\frac{x' + y'}{\sqrt{2}}\right) - 2 = 0$$

$$\frac{(x')^2 - (y')^2}{2} = 2$$

$$\frac{(x')^2}{4} - \frac{(y')^2}{4} = 1$$

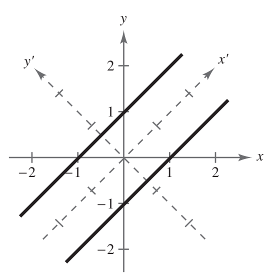


9. $x^2 - 2xy + y^2 - 1 = 0, A = 1, B = -2, C = 1$

$$\cot 2\theta = \frac{A - C}{B} = 0 \implies 2\theta = \frac{\pi}{2} \implies \theta = \frac{\pi}{4}$$

$$x = x'\cos\frac{\pi}{4} - y'\sin\frac{\pi}{4} = x'\left(\frac{\sqrt{2}}{2}\right) - y'\left(\frac{\sqrt{2}}{2}\right) = \frac{x' - y'}{\sqrt{2}}$$

$$y = x'\sin\frac{\pi}{4} + y'\cos\frac{\pi}{4} = x'\left(\frac{\sqrt{2}}{2}\right) + y'\left(\frac{\sqrt{2}}{2}\right) = \frac{x' + y'}{\sqrt{2}}$$



$$x^2 - 2xy + y^2 - 1 = 0$$

$$\left(\frac{x' - y'}{\sqrt{2}}\right)^2 - 2\left(\frac{x' - y'}{\sqrt{2}}\right)\left(\frac{x' + y'}{\sqrt{2}}\right) + \left(\frac{x' + y'}{\sqrt{2}}\right)^2 - 1 = 0$$

$$\frac{(x')^2 - 2(x')(y') + (y')^2}{2} - \frac{2((x')^2 - (y')^2)}{2} + \frac{(x')^2 + 2(x')(y') + (y')^2}{2} - 1 = 0$$

$$2(y')^2 - 1 = 0$$

$$(y')^2 = \frac{1}{2}$$

$$y' = \pm\sqrt{\frac{1}{2}} = \pm\frac{\sqrt{2}}{2}$$

The graph is two parallel lines.

Alternate solution:

$$x^2 - 2xy + y^2 - 1 = 0$$

$$(x - y)^2 = 1$$

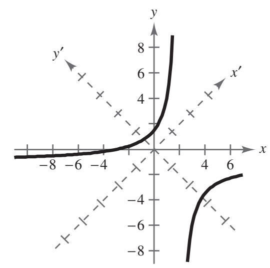
$$x - y = \pm 1$$

$$y = x \pm 1$$

10. $xy + x - 2y + 3 = 0$

$A = 0, B = 1, C = 0$

$$\cot 2\theta = \frac{A - C}{B} = 0 \implies 2\theta = \frac{\pi}{2} \implies \theta = \frac{\pi}{4}$$



$$x = x'\cos\frac{\pi}{4} - y'\sin\frac{\pi}{4} = x'\left(\frac{\sqrt{2}}{2}\right) - y'\left(\frac{\sqrt{2}}{2}\right) = \frac{x' - y'}{\sqrt{2}}$$

$$y = x'\sin\frac{\pi}{4} + y'\cos\frac{\pi}{4} = x'\left(\frac{\sqrt{2}}{2}\right) + y'\left(\frac{\sqrt{2}}{2}\right) = \frac{x' + y'}{\sqrt{2}}$$

$$xy + x - 2y + 3 = 0$$

$$\left(\frac{x' - y'}{\sqrt{2}}\right)\left(\frac{x' + y'}{\sqrt{2}}\right) + \left(\frac{x' - y'}{\sqrt{2}}\right) - 2\left(\frac{x' + y'}{\sqrt{2}}\right) + 3 = 0$$

$$\frac{(x')^2}{2} - \frac{(y')^2}{2} + \frac{x'}{\sqrt{2}} - \frac{y'}{\sqrt{2}} - \frac{2x'}{\sqrt{2}} - \frac{2y'}{\sqrt{2}} + 3 = 0$$

$$\left[(x')^2 - \sqrt{2}x' + \left(\frac{\sqrt{2}}{2}\right)^2\right] - \left[(y')^2 + 3\sqrt{2}y' + \left(\frac{3\sqrt{2}}{2}\right)^2\right] = -6 + \left(\frac{\sqrt{2}}{2}\right)^2 - \left(\frac{3\sqrt{2}}{2}\right)^2$$

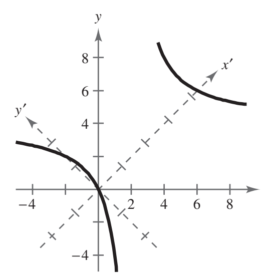
$$\left(x' - \frac{\sqrt{2}}{2}\right)^2 - \left(y' + \frac{3\sqrt{2}}{2}\right)^2 = -10$$

$$\frac{\left(y' + \frac{3\sqrt{2}}{2}\right)^2}{10} - \frac{\left(x' - \frac{\sqrt{2}}{2}\right)^2}{10} = 1$$

11. $xy - 2y - 4x = 0$

$A = 0, B = 1, C = 0$

$$\cot 2\theta = \frac{A - C}{B} = 0 \implies 2\theta = \frac{\pi}{2} \implies \theta = \frac{\pi}{4}$$



$$x = x'\cos\frac{\pi}{4} - y'\sin\frac{\pi}{4} = x'\left(\frac{\sqrt{2}}{2}\right) - y'\left(\frac{\sqrt{2}}{2}\right) = \frac{x' - y'}{\sqrt{2}}$$

$$y = x'\sin\frac{\pi}{4} + y'\cos\frac{\pi}{4} = x'\left(\frac{\sqrt{2}}{2}\right) + y'\left(\frac{\sqrt{2}}{2}\right) = \frac{x' + y'}{\sqrt{2}}$$

$$xy - 2y - 4x = 0$$

$$\left(\frac{x' - y'}{\sqrt{2}}\right)\left(\frac{x' + y'}{\sqrt{2}}\right) - 2\left(\frac{x' + y'}{\sqrt{2}}\right) - 4\left(\frac{x' - y'}{\sqrt{2}}\right) = 0$$

$$\frac{(x')^2}{2} - \frac{(y')^2}{2} - \sqrt{2}x' - \sqrt{2}y' - 2\sqrt{2}x' + 2\sqrt{2}y' = 0$$

$$\left[(x')^2 - 6\sqrt{2}x' + \left(3\sqrt{2}\right)^2\right] - \left[(y')^2 - 2\sqrt{2}y' + \left(\sqrt{2}\right)^2\right] = 0 + \left(3\sqrt{2}\right)^2 - \left(\sqrt{2}\right)^2$$

$$\left(x' - 3\sqrt{2}\right)^2 - \left(y' - \sqrt{2}\right)^2 = 16$$

$$\frac{\left(x' - 3\sqrt{2}\right)^2}{16} - \frac{\left(y' - \sqrt{2}\right)^2}{16} = 1$$

12. $2x^2 - 3xy - 2y^2 + 10 = 0$

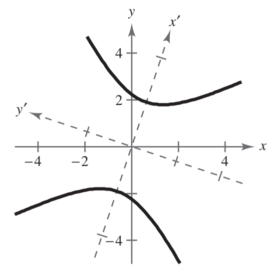
$A = 2, B = -3, C = -2$

$\cot 2\theta = \dfrac{A - C}{B} = -\dfrac{4}{3} \implies \theta \approx 71.57°$

$\cos 2\theta = -\dfrac{4}{5}$

$\sin\theta = \sqrt{\dfrac{1-\cos 2\theta}{2}} = \sqrt{\dfrac{1-(-4/5)}{2}} = \dfrac{3}{\sqrt{10}}$

$\cos\theta = \sqrt{\dfrac{1+\cos 2\theta}{2}} = \sqrt{\dfrac{1+(-4/5)}{2}} = \dfrac{1}{\sqrt{10}}$



$x = x'\cos\theta - y'\sin\theta$
$= x'\left(\dfrac{1}{\sqrt{10}}\right) - y'\left(\dfrac{3}{\sqrt{10}}\right)$
$= \dfrac{x' - 3y'}{\sqrt{10}}$

$y = x'\sin\theta + y'\cos\theta$
$= x'\left(\dfrac{3}{\sqrt{10}}\right) + y'\left(\dfrac{1}{\sqrt{10}}\right)$
$= \dfrac{3x' + y'}{\sqrt{10}}$

$$2x^2 - 3xy - 2y^2 + 10 = 0$$

$$2\left(\frac{x'-3y'}{\sqrt{10}}\right)^2 - 3\left(\frac{x'-3y'}{\sqrt{10}}\right)\left(\frac{3x'+y'}{\sqrt{10}}\right) - 2\left(\frac{3x'+y'}{\sqrt{10}}\right)^2 + 10 = 0$$

$$\frac{(x')^2}{5} - \frac{6x'y'}{5} + \frac{9(y')^2}{5} - \frac{9(x')^2}{10} + \frac{24x'y'}{10} + \frac{9(y')^2}{10} - \frac{9(x')^2}{5} - \frac{6x'y'}{5} - \frac{(y')^2}{5} + 10 = 0$$

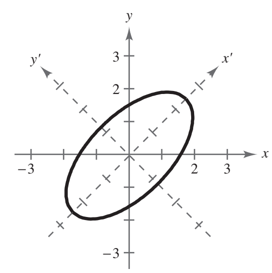
$$-\frac{5}{2}(x')^2 + \frac{5}{2}(y')^2 = -10$$

$$\frac{(x')^2}{4} - \frac{(y')^2}{4} = 1$$

13. $5x^2 - 6xy + 5y^2 - 12 = 0$

$A = 5, B = -6, C = 5$

$\cot 2\theta = \dfrac{A - C}{B} = 0 \implies 2\theta = \dfrac{\pi}{2} \implies \theta = \dfrac{\pi}{4}$



$x = x'\cos\dfrac{\pi}{4} - y'\sin\dfrac{\pi}{4}$
$= x'\left(\dfrac{\sqrt{2}}{2}\right) - y'\left(\dfrac{\sqrt{2}}{2}\right)$
$= \dfrac{x' - y'}{\sqrt{2}}$

$y = x'\sin\dfrac{\pi}{4} + y'\cos\dfrac{\pi}{4}$
$= x'\left(\dfrac{\sqrt{2}}{2}\right) + y'\left(\dfrac{\sqrt{2}}{2}\right)$
$= \dfrac{x' + y'}{\sqrt{2}}$

$$5x^2 - 6xy + 5y^2 - 12 = 0$$

$$5\left(\frac{x'-y'}{\sqrt{2}}\right)^2 - 6\left(\frac{x'-y'}{\sqrt{2}}\right)\left(\frac{x'+y'}{\sqrt{2}}\right) + 5\left(\frac{x'+y'}{\sqrt{2}}\right)^2 - 12 = 0$$

$$\frac{5(x')^2}{2} - 5x'y' + \frac{5(y')^2}{2} - 3(x')^2 + 3(y')^2 + \frac{5(x')^2}{2} + 5x'y' + \frac{5(y')^2}{2} - 12 = 0$$

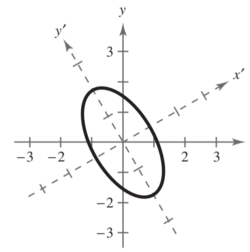
$$2(x')^2 + 8(y')^2 = 12$$

$$\frac{(x')^2}{6} + \frac{(y')^2}{3/2} = 1$$

14. $13x^2 + 6\sqrt{3}xy + 7y^2 - 16 = 0$

$A = 13, B = 6\sqrt{3}, C = 7$

$$\cot 2\theta = \frac{A - C}{B} = \frac{1}{\sqrt{3}} \implies 2\theta = \frac{\pi}{3} \implies \theta = \frac{\pi}{6}$$



$$x = x'\cos\frac{\pi}{6} - y'\sin\frac{\pi}{6} = x'\left(\frac{\sqrt{3}}{2}\right) - y'\left(\frac{1}{2}\right) = \frac{\sqrt{3}x' - y'}{2}$$

$$y = x'\sin\frac{\pi}{6} + y'\cos\frac{\pi}{6} = x'\left(\frac{1}{2}\right) + y'\left(\frac{\sqrt{3}}{2}\right) = \frac{x' + \sqrt{3}y'}{2}$$

$$13x^2 + 6\sqrt{3}xy + 7y^2 - 16 = 0$$

$$13\left(\frac{\sqrt{3}x' - y'}{2}\right)^2 + 6\sqrt{3}\left(\frac{\sqrt{3}x' - y'}{2}\right)\left(\frac{x' + \sqrt{3}y'}{2}\right) + 7\left(\frac{x' + \sqrt{3}y'}{2}\right)^2 - 16 = 0$$

$$\frac{39(x')^2}{4} - \frac{13\sqrt{3}x'y'}{2} + \frac{13(y')^2}{4} + \frac{18(x')^2}{4} + \frac{18\sqrt{3}x'y'}{4} - \frac{6\sqrt{3}x'y'}{4}$$

$$-\frac{18(y')^2}{4} + \frac{7(x')^2}{4} + \frac{7\sqrt{3}x'y'}{2} + \frac{21(y')^2}{4} - 16 = 0$$

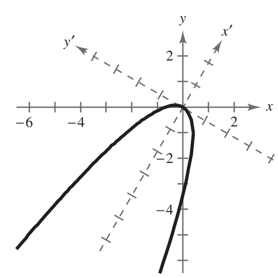
$$16(x')^2 + 4(y')^2 = 16$$

$$\frac{(x')^2}{1} + \frac{(y')^2}{4} = 1$$

15. $3x^2 - 2\sqrt{3}xy + y^2 + 2x + 2\sqrt{3}y = 0$

$A = 3, B = -2\sqrt{3}, C = 1$

$$\cot 2\theta = \frac{A - C}{B} = -\frac{1}{\sqrt{3}} \implies \theta = 60^\circ$$



$$x = x'\cos 60^\circ - y'\sin 60^\circ = x'\left(\frac{1}{2}\right) - y'\left(\frac{\sqrt{3}}{2}\right) = \frac{x' - \sqrt{3}y'}{2}$$

$$y = x'\sin 60^\circ + y'\cos 60^\circ = x'\left(\frac{\sqrt{3}}{2}\right) + y'\left(\frac{1}{2}\right) = \frac{\sqrt{3}x' + y'}{2}$$

$$3x^2 - 2\sqrt{3}xy + y^2 + 2x + 2\sqrt{3}y = 0$$

$$3\left(\frac{x' - \sqrt{3}y'}{2}\right)^2 - 2\sqrt{3}\left(\frac{x' - \sqrt{3}y'}{2}\right)\left(\frac{\sqrt{3}x' + y'}{2}\right) + \left(\frac{\sqrt{3}x' + y'}{2}\right)^2 + 2\left(\frac{x' - \sqrt{3}y'}{2}\right) + 2\sqrt{3}\left(\frac{\sqrt{3}x' + y'}{2}\right) = 0$$

$$\frac{3(x')^2}{4} - \frac{6\sqrt{3}x'y'}{4} + \frac{9(y')^2}{4} - \frac{6(x')^2}{4} + \frac{4\sqrt{3}x'y'}{4} + \frac{6(y')^2}{4} + \frac{3(x')^2}{4} + \frac{2\sqrt{3}x'y'}{4} + \frac{(y')^2}{4}$$

$$+ x' - \sqrt{3}y' + 3x' + \sqrt{3}y' = 0$$

$$4(y')^2 + 4x' = 0$$

$$(y')^2 = -x'$$

16. $16x^2 - 24xy + 9y^2 - 60x - 80y + 100 = 0$

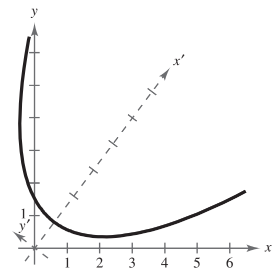
$A = 16, B = -24, C = 9$

$\cot 2\theta = \dfrac{A - C}{B} = -\dfrac{7}{24} \implies \theta \approx 53.13°$

$\cos 2\theta = -\dfrac{7}{25}$

$\sin\theta = \sqrt{\dfrac{1 - \cos 2\theta}{2}} = \sqrt{\dfrac{1 - (-7/25)}{2}} = \dfrac{4}{5}$

$\cos\theta = \sqrt{\dfrac{1 + \cos 2\theta}{2}} = \sqrt{\dfrac{1 + (-7/25)}{2}} = \dfrac{3}{5}$



$x = x'\cos\theta - y'\sin\theta$

$= x'\left(\dfrac{3}{5}\right) - y'\left(\dfrac{4}{5}\right)$

$y = x'\sin\theta + y'\cos\theta$

$= x'\left(\dfrac{4}{3}\right) + y'\left(\dfrac{3}{5}\right) = \dfrac{4x' + 3y'}{5}$

$$16x^2 - 24xy + 9y^2 - 60x - 80y + 100 = 0$$

$$16\left(\frac{3x' - 4y'}{5}\right)^2 - 24\left(\frac{3x' - 4y'}{5}\right)\left(\frac{4x' + 3y'}{5}\right) + 9\left(\frac{4x' + 3y'}{5}\right)^2 - 60\left(\frac{3x' - 4y'}{5}\right) - 80\left(\frac{4x' + 3y'}{5}\right) + 100 = 0$$

$$\frac{144(x')^2}{25} - \frac{384x'y'}{25} + \frac{256(y')^2}{25} - \frac{288(x')^2}{25} + \frac{168x'y'}{25} + \frac{288(y')^2}{25} + \frac{144(x')^2}{25} + \frac{216x'y'}{25}$$

$$+ \frac{81(y')^2}{25} - 36x' + 48y' - 64x' - 48y' + 100 = 0$$

$$25(y')^2 - 100x' + 100 = 0$$

$$(y')^2 = 4(x' - 1)$$

17. $9x^2 + 24xy + 16y^2 + 90x - 130y = 0$

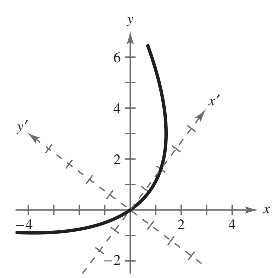
$A = 9, B = 24, C = 16$

$\cot 2\theta = \dfrac{A - C}{B} = -\dfrac{7}{24} \implies \theta \approx 53.13°$

$\cos 2\theta = -\dfrac{7}{25}$

$\sin\theta = \sqrt{\dfrac{1 - \cos 2\theta}{2}} = \sqrt{\dfrac{1 - \left(-\frac{7}{25}\right)}{2}} = \dfrac{4}{5}$

$\cos\theta = \sqrt{\dfrac{1 + \cos 2\theta}{2}} = \sqrt{\dfrac{1 + \left(-\frac{7}{25}\right)}{2}} = \dfrac{3}{5}$



$x = x'\cos\theta - y'\sin\theta$

$= x'\left(\dfrac{3}{5}\right) - y'\left(\dfrac{4}{5}\right) = \dfrac{3x' - 4y'}{5}$

$y = x'\sin\theta + y'\cos\theta$

$= x'\left(\dfrac{4}{5}\right) + y'\left(\dfrac{3}{5}\right)$

$= \dfrac{4x' + 3y'}{5}$

—CONTINUED—

17. —CONTINUED—

$$9x^2 + 24xy + 16y^2 + 90x - 130y = 0$$

$$9\left(\frac{3x' - 4y'}{5}\right)^2 + 24\left(\frac{3x' - 4y'}{5}\right)\left(\frac{4x' + 3y'}{5}\right) + 16\left(\frac{4x' + 3y'}{5}\right)^2 + 90\left(\frac{3x' - 4y'}{5}\right) - 130\left(\frac{4x' + 3y'}{5}\right) = 0$$

$$\frac{81(x')^2}{25} - \frac{216x'y'}{25} + \frac{144(y')^2}{25} + \frac{288(x')^2}{25} - \frac{168x'y'}{25} - \frac{288(y')^2}{25} + \frac{256(x')^2}{25} + \frac{384x'y'}{25} + \frac{144(y')^2}{25}$$

$$+ 54x' - 72y' - 104x' - 78y' = 0$$

$$25(x')^2 - 50x' - 150y' = 0$$

$$(x')^2 - 2x' = 6y'$$

$$(x')^2 - 2x' + 1 = 6y' + 1$$

$$(x' - 1)^2 = 6\left(y' + \frac{1}{6}\right)$$

18. $9x^2 + 24xy + 16y^2 + 80x - 60y = 0$

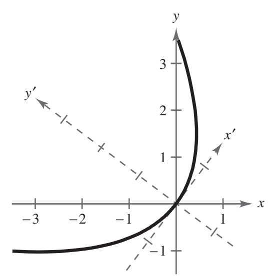
$A = 9, B = 24, C = 16$

$$\cot 2\theta = \frac{A - C}{B} = -\frac{7}{24} \implies \theta \approx 53.13°$$

$$\cos 2\theta = -\frac{7}{25}$$

$$\sin\theta = \sqrt{\frac{1 - \cos 2\theta}{2}} = \sqrt{\frac{1 - (-7/25)}{2}} = \frac{4}{5}$$

$$\cos\theta = \sqrt{\frac{1 + \cos 2\theta}{2}} = \sqrt{\frac{1 + (-7/25)}{2}} = \frac{3}{5}$$



$$x = x'\cos\theta - y'\sin\theta \qquad y = x'\sin\theta + y'\cos\theta$$

$$= x'\left(\frac{3}{5}\right) - y'\left(\frac{4}{5}\right) \qquad = x'\left(\frac{4}{5}\right) + y'\left(\frac{3}{5}\right)$$

$$= \frac{3x' - 4y'}{5} \qquad = \frac{4x' + 3y}{5}$$

$$9x^2 + 24xy + 16y^2 + 80x - 60y = 0$$

$$9\left(\frac{3x' - 4y'}{5}\right)^2 + 24\left(\frac{3x' - 4y'}{5}\right)\left(\frac{4x' + 3y'}{5}\right) + 16\left(\frac{4x' + 3y'}{5}\right)^2 + 80\left(\frac{3x' - 4y'}{5}\right) - 60\left(\frac{4x' + 3y'}{5}\right) = 0$$

$$\frac{81(x')^2}{25} - \frac{216x'y'}{25} + \frac{144(y')^2}{25} + \frac{288(x')^2}{25} - \frac{168x'y'}{25} - \frac{288(y')^2}{25} + \frac{256(x')^2}{25} + \frac{384x'y'}{25} + \frac{144(y')^2}{25}$$

$$+ 48x' - 64x' - 48x' - 36x' = 0$$

$$25(x')^2 - 100y' = 0$$

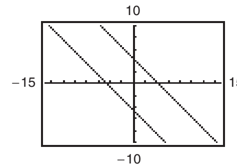
$$(x')^2 = 4y'$$

$$\frac{1}{4}(x')^2 = y'$$

19. $x^2 + 2xy + y^2 = 20$

$A = 1, B = 2, C = 1$

$\cot 2\theta = \dfrac{A - C}{B} = \dfrac{1 - 1}{2} = 0 \implies \theta = \dfrac{\pi}{4}$ or $45°$



To graph the conic using a graphing calculator, we need to solve for y in terms of x.

$$(x + y)^2 = 20$$
$$x + y = \pm\sqrt{20}$$
$$y = -x \pm \sqrt{20}$$

Use $y_1 = -x + \sqrt{20}$ and $y_2 = -x - \sqrt{20}$.

20. $x^2 - 4xy + 2y^2 = 6$

$A = 1, B = -4, C = 2$

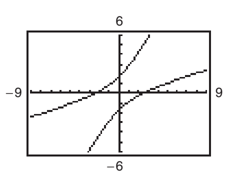
$\cot 2\theta = \dfrac{A - C}{B} = \dfrac{1 - 2}{-4} = \dfrac{1}{4}$

$$\frac{1}{\tan 2\theta} = \frac{1}{4}$$
$$\tan 2\theta = 4$$
$$2\theta \approx 75.96$$
$$\theta \approx 37.98°$$

To graph conic with a graphing calculator, we need to solve for y in terms of x.

$$x^2 - 4xy + 2y^2 = 6$$
$$y^2 - 2xy + x^2 = 3 - \frac{x^2}{2} + x^2$$
$$(y - x)^2 = 3 + \frac{x^2}{2}$$
$$y - x = \pm\sqrt{3 + \frac{x^2}{2}}$$
$$y = x \pm \sqrt{3 + \frac{x^2}{2}}$$

Enter $y_1 = x + \sqrt{3 + \dfrac{x^2}{2}}$ and $y_2 = x - \sqrt{3 + \dfrac{x^2}{2}}$.



21. $17x^2 + 32xy - 7y^2 = 75$

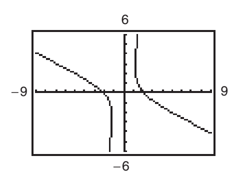
$\cot 2\theta = \dfrac{A - C}{B} = \dfrac{17 + 7}{32} = \dfrac{24}{32} = \dfrac{3}{4} \implies \theta \approx 26.57°$

Solve for y in terms of x by completing the square.

$$-7y^2 + 32xy = -17x^2 + 75$$
$$y^2 - \frac{32}{7}xy = \frac{17}{7}x^2 - \frac{75}{7}$$
$$y^2 - \frac{32}{7}xy + \frac{256}{49}x^2 = \frac{119}{49}x^2 - \frac{525}{49} + \frac{256}{49}x^2$$
$$\left(y - \frac{16}{7}x\right)^2 = \frac{375x^2 - 525}{49}$$
$$y = \frac{16}{7}x \pm \sqrt{\frac{375x^2 - 525}{49}}$$
$$y = \frac{16x \pm 5\sqrt{15x^2 - 21}}{7}$$

Use $y_1 = \dfrac{16x + 5\sqrt{15x^2 - 21}}{7}$

and $y_2 = \dfrac{16x - 5\sqrt{15x^2 - 21}}{7}$.



22. $40x^2 + 36xy + 25y^2 = 52$

$A = 40, B = 36, C = 25$

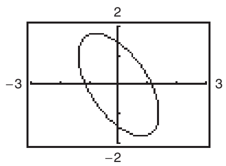
$$\cot 2\theta = \frac{A - C}{B} = \frac{40 - 25}{36} = \frac{5}{12}$$

$$\frac{1}{\tan 2\theta} = \frac{5}{12}$$

$$\tan 2\theta = \frac{12}{5}$$

$$2\theta \approx 67.38°$$

$$\theta \approx 33.69°$$



Solve for y in terms of x by completing the square:

$$25y^2 + 36xy = 52 - 40x^2$$

$$y^2 + \frac{36}{25}xy = \frac{52}{25} - \frac{40}{25}x^2$$

$$y^2 + \frac{36}{25}xy + \frac{324}{625}x^2 = \frac{52}{25} - \frac{40}{25}x^2 + \frac{324}{625}x^2$$

$$\left(y + \frac{18}{25}x\right)^2 = \frac{1300 - 676x^2}{625}$$

$$y + \frac{18}{25}x = \pm\sqrt{\frac{1300 - 676x^2}{625}}$$

$$y = \frac{-18x \pm \sqrt{1300 - 676x^2}}{25}$$

Enter $y_1 = \dfrac{-18x + \sqrt{1300 - 676x^2}}{25}$ and

$y_2 = \dfrac{-18x - \sqrt{1300 - 676x^2}}{25}$.

23. $32x^2 + 48xy + 8y^2 = 50$

$$\cot 2\theta = \frac{A - C}{B} = \frac{24}{48} = \frac{1}{2} \implies \theta \approx 31.72°$$

Solve for y in terms of x by completing the square.

$$8y^2 + 48xy = -32x^2 + 50$$

$$y^2 + 6xy = -4x^2 + \frac{25}{4}$$

$$y^2 + 6xy + 9x^2 = -4x^2 + \frac{25}{4} + 9x^2$$

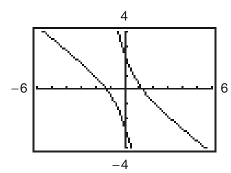
$$(y + 3x)^2 = 5x^2 + \frac{25}{4}$$

$$y + 3x = \pm\sqrt{5x^2 + \frac{25}{4}}$$

$$y = -3x \pm \sqrt{5x^2 + \frac{25}{4}}$$

Use $y_1 = -3x + \sqrt{5x^2 + \dfrac{25}{4}}$ and

$y_2 = -3x - \sqrt{5x^2 + \dfrac{25}{4}}$.



24. $24x^2 + 18xy + 12y^2 = 34$

$A = 24, B = 18, C = 12$

$$\cot 2\theta = \frac{A - C}{B} = \frac{24 - 12}{18} = \frac{2}{3}$$

$$\tan 2\theta = \frac{3}{2}$$

$$2\theta \approx 56.31°$$

$$\theta \approx 28.15°$$

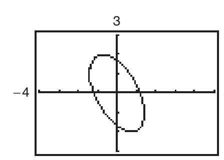
Solve for y in terms of x by completing the square:

$$12x^2 + 9xy + 6y^2 = 17$$

$$6\left(y^2 + \frac{3}{2}xy + \frac{9}{16}x^2\right) = 17 - 12x^2 + \frac{27}{8}x^2 = 17 - \frac{69}{8}x^2$$

$$\left(y + \frac{3}{4}x\right)^2 = \frac{136 - 69x^2}{48}$$

$$y = -\frac{3}{4}x \pm \sqrt{\frac{136 - 69x^2}{48}} = \frac{-9x \pm \sqrt{3(136 - 69x^2)}}{12}$$



Enter $y_1 = \dfrac{-9x + \sqrt{3(136 - 69x^2)}}{12}$ and $y_2 = \dfrac{-9x - \sqrt{3(136 - 69x^2)}}{12}$.

25. $4x^2 - 12xy + 9y^2 + (4\sqrt{13} - 12)x - (6\sqrt{13} + 8)y = 91$

$A = 4, B = -12, C = 9$

$$\cot 2\theta = \frac{A - C}{B} = \frac{4 - 9}{-12} = \frac{5}{12}$$

$$\frac{1}{\tan 2\theta} = \frac{5}{12}$$

$$\tan 2\theta = \frac{12}{5}$$

$$2\theta \approx 67.38°$$

$$\theta \approx 33.69°$$

Solve for y in terms of x with the quadratic formula:

$$4x^2 - 12xy + 9y^2 + (4\sqrt{13} - 12)x - (6\sqrt{13} + 8)y = 91$$

$$9y^2 - (12x + 6\sqrt{13} + 8)y + (4x^2 + 4\sqrt{13}x - 12x - 91) = 0$$

$$a = 9, b = -(12x + 6\sqrt{13} + 8), c = 4x^2 + 4\sqrt{13}x - 12x - 91$$

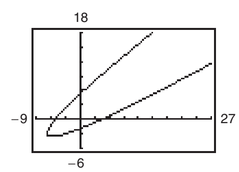
$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$y = \frac{(12x + 6\sqrt{13} + 8) \pm \sqrt{(12x + 6\sqrt{13} + 8)^2 - 4(9)(4x^2 + 4\sqrt{13}x - 12x - 91)}}{18}$$

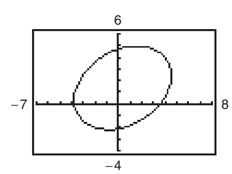
$$= \frac{(12x + 6\sqrt{13} + 8) \pm \sqrt{624x + 3808 + 96\sqrt{13}}}{18}$$

Enter $y_1 = \dfrac{12x + 6\sqrt{13} + 8 + \sqrt{624x + 3808 + 96\sqrt{13}}}{18}$

and $y_2 = \dfrac{12x + 6\sqrt{13} + 8 - \sqrt{624x + 3808 + 96\sqrt{13}}}{18}$.



26. $6x^2 - 4xy + 8y^2 + (5\sqrt{5} - 10)x - (7\sqrt{5} + 5)y = 80$



$A = 6, B = -4, C = 8$

$$\cot 2\theta = \frac{A - C}{B} = \frac{6 - 8}{-4} = \frac{1}{2}$$

$$\tan 2\theta = 2$$

$$2\theta \approx 63.43°$$

$$\theta \approx 31.72°$$

Solve for y in terms of x using the quadratic formula.

$$8y^2 - (4x + 7\sqrt{5} + 5)y + 6x^2 + (5\sqrt{5} - 10)x - 80 = 0$$

$$y = \frac{1}{16}\left[4x + 7\sqrt{5} + 5 \pm \sqrt{(4x + 7\sqrt{5} + 5)^2 - 32(6x^2 + (5\sqrt{5} - 10)x - 80)}\right]$$

Enter y_1 and y_2 from the above expression.

27. $xy + 2 = 0$

$B^2 - 4AC = 1 \implies$ The graph is a hyperbola.

$$\cot 2\theta = \frac{A - C}{B} = 0 \implies \theta = 45°$$

Matches graph (e).

28. $x^2 + 2xy + y^2 = 0$

$$(x + y)^2 = 0$$

$$x + y = 0$$

$$y = -x$$

The graph is a line. Matches graph (f).

29. $-2x^2 + 3xy + 2y^2 + 3 = 0$

$B^2 - 4AC = (3)^2 - 4(-2)(2) = 25 \implies$

The graph is a hyperbola.

$\cot 2\theta = \dfrac{A - C}{B} = -\dfrac{4}{3} \implies \theta \approx -18.43°$

Matches graph (b).

30. $x^2 - xy + 3y^2 - 5 = 0$

$A = 1, B = -1, C = 3$

$B^2 - 4AC = (-1)^2 - 4(1)(3) = -11$

The graph is an ellipse.

$\cot 2\theta = \dfrac{A - C}{B} = \dfrac{1 - 3}{-1} = 2 \implies \theta \approx 13.28°$

Matches graph (a).

31. $3x^2 + 2xy + y^2 - 10 = 0$

$B^2 - 4AC = (2)^2 - 4(3)(1) = -8 \implies$

The graph is an ellipse or circle.

$\cot 2\theta = \dfrac{A - C}{B} = 1 \implies \theta = 22.5°$

Matches graph (d).

32. $x^2 - 4xy + 4y^2 + 10x - 30 = 0$

$A = 1, B = -4, C = 4$

$B^2 - 4AC = (-4)^2 - 4(1)(4) = 0$

The graph is a parabola.

$\cot 2\theta = \dfrac{A - C}{B} = \dfrac{1 - 4}{-4} = \dfrac{3}{4} \implies \theta \approx 26.57°$

Matches graph (c).

33. (a) $16x^2 - 8xy + y^2 - 10x + 5y = 0$

$B^2 - 4AC = (-8)^2 - 4(16)(1) = 0$

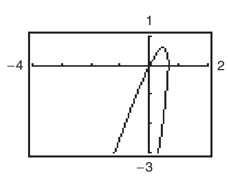
The graph is a parabola.

(b) $y^2 + (-8x + 5)y + (16x^2 - 10x) = 0$

$$y = \frac{-(-8x + 5) \pm \sqrt{(-8x + 5)^2 - 4(1)(16x^2 - 10x)}}{2(1)}$$

$$= \frac{(8x - 5) \pm \sqrt{(8x - 5)^2 - 4(16x^2 - 10x)}}{2}$$

(c)



34. (a) $x^2 - 4xy - 2y^2 - 6 = 0$

$A = 1, B = -4, C = -2$

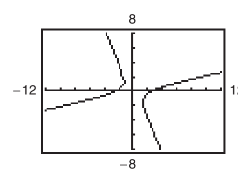
$B^2 - 4AC = (-4)^2 - 4(1)(-2) = 24 > 0$

The graph is a hyperbola.

(b) $-2y^2 - 4xy + x^2 - 6 = 0$

$$y = -\frac{1}{4}\left[4x \pm \sqrt{16x^2 + 8(x^2 - 6)}\right]$$

(c)



35. (a) $12x^2 - 6xy + 7y^2 - 45 = 0$

$B^2 - 4AC = (-6)^2 - 4(12)(7) = -300 < 0$

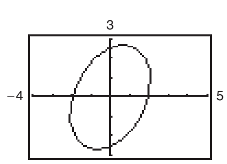
The graph is an ellipse.

(b) $7y^2 + (-6x)y + (12x^2 - 45) = 0$

$$y = \frac{-(-6x) \pm \sqrt{(-6x)^2 - 4(7)(12x^2 - 45)}}{2(7)}$$

$$= \frac{6x \pm \sqrt{36x^2 - 28(12x^2 - 45)}}{14}$$

(c)



36. (a) $2x^2 + 4xy + 5y^2 + 3x - 4y - 20 = 0$

$A = 2, B = 4, C = 5$

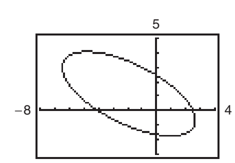
$B^2 - 4AC = 4^2 - 4(2)(5) = 16 - 40 = -24 < 0$

The graph is an ellipse.

(b) $5y^2 + (4x - 4)y + 2x^2 + 3x - 20 = 0$

$$y = \frac{1}{10}\left[-(4x - 4) \pm \sqrt{(4x - 4)^2 - 20(2x^2 + 3x - 20)}\right]$$

(c)



37. (a) $x^2 - 6xy - 5y^2 + 4x - 22 = 0$

$B^2 - 4AC = (-6)^2 - 4(1)(-5) = 56 > 0$

The graph is a hyperbola.

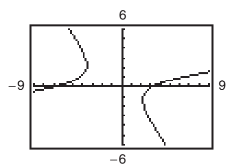
(b) $-5y^2 + (-6x)y + (x^2 + 4x - 22) = 0$

$$y = \frac{-(-6x) \pm \sqrt{(-6x)^2 - 4(-5)(x^2 + 4x - 22)}}{2(-5)}$$

$$= \frac{6x \pm \sqrt{36x^2 + 20(x^2 + 4x - 22)}}{-10}$$

$$= \frac{-6x \pm \sqrt{36x^2 + 20(x^2 + 4x - 22)}}{10}$$

(c)



38. (a) $36x^2 - 60xy + 25y^2 + 9y = 0$

$A = 36, B = -60, C = 25$

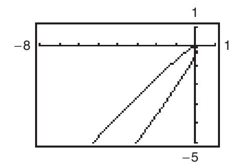
$B^2 - 4AC = (-60)^2 - 4(36)(25) = 0$

The graph is a parabola.

(b) $25y^2 - (60x - 9)y + 36x^2 = 0$

$$y = \frac{1}{50}\left[60x - 9 \pm \sqrt{(60x - 9)^2 - 3600x^2}\right]$$

(c)



39. (a) $x^2 + 4xy + 4y^2 - 5x - y - 3 = 0$

$B^2 - 4AC = (4)^2 - 4(1)(4) = 0$

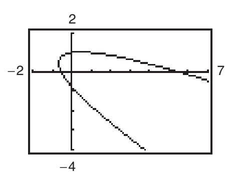
The graph is a parabola.

(b) $4y^2 + (4x - 1)y + (x^2 - 5x - 3) = 0$

$$y = \frac{-(4x - 1) \pm \sqrt{(4x - 1)^2 - 4(4)(x^2 - 5x - 3)}}{2(4)}$$

$$= \frac{-(4x - 1) \pm \sqrt{(4x - 1)^2 - 16(x^2 - 5x - 3)}}{8}$$

(c)



40. (a) $x^2 + xy + 4y^2 + x + y - 4 = 0$

$A = 1, B = 1, C = 4$

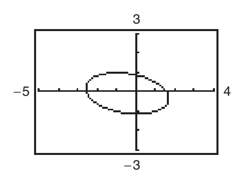
$B^2 - 4AC = 1^2 - 4(1)(4) = -15$

The graph is an ellipse.

(b) $4y^2 + (x + 1)y + x^2 + x - 4 = 0$

$$y = \frac{1}{8}\left[-(x + 1) \pm \sqrt{(x + 1)^2 - 16(x^2 + x - 4)}\right]$$

(c)

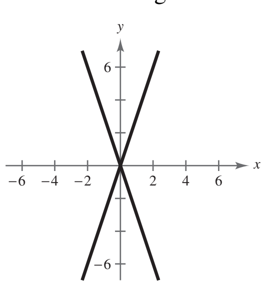


41. $y^2 - 9x^2 = 0$

$y^2 = 9x^2$

$y = \pm 3x$

Two intersecting lines

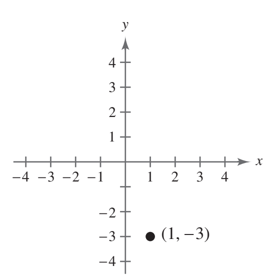


42. $x^2 + y^2 - 2x + 6y + 10 = 0$

$(x^2 - 2x + 1) + (y^2 + 6y + 9) = -10 + 1 + 9$

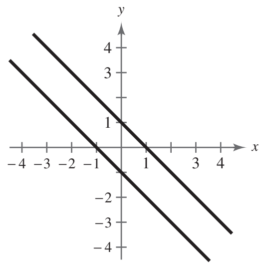
$(x - 1)^2 + (y + 3)^2 = 0$

Point at $(1, -3)$

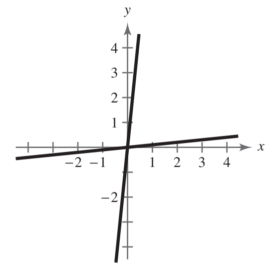


43.
$$\begin{aligned} x^2 + 2xy + y^2 - 1 &= 0 \\ (x+y)^2 - 1 &= 0 \\ (x+y)^2 &= 1 \\ x + y &= \pm 1 \\ y &= -x \pm 1 \end{aligned}$$

Two parallel lines



44.
$$\begin{aligned} x^2 - 10xy + y^2 &= 0 \\ y^2 - 10xy + 25x^2 &= 25x^2 - x^2 \\ (y - 5x)^2 &= 24x^2 \\ y - 5x &= \pm\sqrt{24x^2} \\ y &= 5x \pm 2\sqrt{6}x \\ y &= \left(5 \pm 2\sqrt{6}\right)x \end{aligned}$$

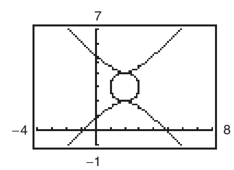


45.
$$\begin{aligned} -x^2 + y^2 + 4x - 6y + 4 &= 0 \implies (y-3)^2 - (x-2)^2 = 1 \\ x^2 + y^2 - 4x - 6y + 12 &= 0 \implies (x-2)^2 + (y-3)^2 = 1 \\ \hline 2y^2 - 12y + 16 &= 0 \\ 2(y-2)(y-4) &= 0 \\ y = 2 \text{ or } y &= 4 \end{aligned}$$

For $y = 2$: $x^2 + 2^2 - 4x - 6(2) + 12 = 0$
$$\begin{aligned} x^2 - 4x + 4 &= 0 \\ (x-2)^2 &= 0 \\ x &= 2 \end{aligned}$$

For $y = 4$: $x^2 + 4^2 - 4x - 6(4) + 12 = 0$
$$\begin{aligned} x^2 - 4x + 4 &= 0 \\ (x-2)^2 &= 0 \\ x &= 2 \end{aligned}$$

The points of intersection are $(2, 2)$ and $(2, 4)$.



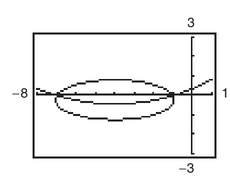
46.
$$\begin{aligned} -x^2 - y^2 - 8x + 20y - 7 &= 0 \implies (x+4)^2 + (y-10)^2 = 109 \\ x^2 + 9y^2 + 8x + 4y + 7 &= 0 \implies (x+4)^2 + 9\left(y + \tfrac{2}{9}\right)^2 = \tfrac{85}{9} \\ \hline 8y^2 + 24y &= 0 \\ 8y(y+3) &= 0 \\ y = 0 \text{ or } y &= -3 \end{aligned}$$

When $y = 0$: $x^2 + 9(0)^2 + 8x + 4(0) + 7 = 0$
$$\begin{aligned} (x+7)(x+1) &= 0 \\ x &= -7, -1 \end{aligned}$$

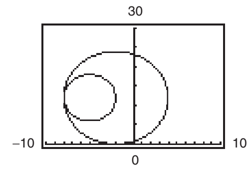
When $y = -3$: $x^2 + 9(-3)^2 + 8x + 4(-3) + 7 = 0$
$$x^2 + 8x + 76 = 0$$

No real solution

Points of intersection: $(-7, 0), (-1, 0)$



47. $$\begin{array}{r} -4x^2 - y^2 - 16x + 24y - 16 = 0 \\ \underline{4x^2 + y^2 + 40x - 24y + 208 = 0} \\ 24x \qquad\quad + 192 = 0 \\ x = -8 \end{array}$$



When $x = -8$: $4(-8)^2 + y^2 + 40(-8) - 24y + 208 = 0$

$$y^2 - 24y + 144 = 0$$

$$(y - 12)^2 = 0$$

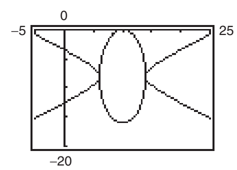
$$y = 12$$

The point of intersection is $(-8, 12)$. In standard form the equations are:

$$\frac{(x + 2)^2}{36} + \frac{(y - 12)^2}{144} = 1 \text{ and } \frac{(x + 5)^2}{9} + \frac{(y - 12)^2}{36} = 1$$

48. $$\begin{array}{rl} x^2 - 4y^2 - 20x - 64y - 172 = 0 & \Longrightarrow \quad (x - 10)^2 - 4(y + 8)^2 = 16 \\ \underline{16x^2 + 4y^2 - 320x + 64y + 1600 = 0} & \Longrightarrow 16(x - 10)^2 + 4(y + 8)^2 = 256 \\ 17x^2 \qquad - 340x \qquad 1428 = 0 & \\ (17x - 238)(x - 6) = 0 & \\ x = 6 \text{ or } x = 14 & \end{array}$$

When $x = 6$: $6^2 - 4y^2 - 20(6) - 64y - 172 = 0$



$$-4y^2 - 64y - 256 = 0$$

$$y^2 + 16y + 64 = 0$$

$$(y + 8)^2 = 0$$

$$y = -8$$

When $x = 14$: $14^2 - 4y^2 - 20(14) - 64y - 172 = 0$

$$-4y^2 - 64y - 256 = 0$$

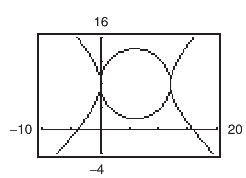
$$y^2 + 16y + 64 = 0$$

$$(y + 8)^2 = 0$$

$$y = -8$$

Points of intersection: $(6, -8), (14, -8)$

49. $$\begin{array}{r} x^2 - y^2 - 12x + 16y - 64 = 0 \\ \underline{x^2 + y^2 - 12x - 16y + 64 = 0} \\ 2x^2 \qquad - 24x \qquad\qquad = 0 \\ 2x(x - 12) = 0 \\ x = 0 \quad \text{or} \quad x = 12 \end{array}$$



When $x = 0$: $0^2 + y^2 - 12(0) - 16y + 64 = 0$

$$y^2 - 16y + 64 = 0$$

$$(y - 8)^2 = 0$$

$$y = 8$$

When $x = 12$: $12^2 + y^2 - 12(12) - 16y + 64 = 0$

$$y^2 - 16y + 64 = 0$$

$$(y - 8)^2 = 0$$

$$y = 8$$

The points of intersection are $(0, 8)$ and $(12, 8)$. The standard forms of the equations are:

$$\frac{(x - 6)^2}{36} - \frac{(y - 8)^2}{36} = 1 \text{ and } (x - 6)^2 + (y - 8)^2 = 36$$

50.

$$\begin{array}{rl} x^2 + 4y^2 - 2x - 8y + 1 = 0 & \Longrightarrow (x-1)^2 + 4(y-1)^2 = 4 \\ \underline{-x^2 \qquad\quad + 2x - 4y - 1 = 0} & \Longrightarrow y = -\frac{1}{4}(x-1)^2 \\ 4y^2 \qquad -12y \qquad = 0 & \\ 4y(y-3) = 0 & \\ y = 0 \text{ or } y = 3 & \end{array}$$

When $y = 0$: $x^2 + 4(0)^2 - 2x - 8(0) + 1 = 0$

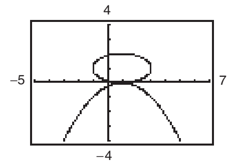
$$x^2 - 2x + 1 = 0$$
$$(x-1)^2 = 0$$
$$x = 1$$

When $y = 3$: $-x^2 + 2x - 4(3) - 1 = 0$

$$x^2 - 2x + 13 = 0$$

No real solution

Point of intersection: $(1, 0)$



51.

$$\begin{array}{r} -16x^2 - \quad y^2 + 24y - \quad 80 = 0 \\ \underline{16x^2 + 25y^2 \qquad\quad - 400 = 0} \\ 24y^2 + 24y - 480 = 0 \\ 24(y+5)(y-4) = 0 \\ y = -5 \text{ or } y = 4 \end{array}$$

When $y = -5$: $16x^2 + 25(-5)^2 - 400 = 0$

$$16x^2 = -225$$

No real solution

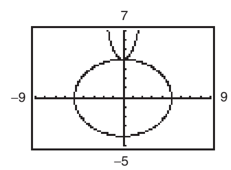
When $y = 4$: $16x^2 + 25(4)^2 - 400 = 0$

$$16x^2 = 0$$
$$x = 0$$

The point of intersection is (0, 4).
In standard form the equations are:

$$\frac{x^2}{4} + \frac{(y-12)^2}{64} = 1$$

$$\frac{x^2}{25} + \frac{y^2}{16} = 1$$



52.

$$\begin{array}{rl} 16x^2 - y^2 \qquad + 16y - 128 = 0 & \Longrightarrow 16x^2 - (y-8)^2 = 64 \\ \underline{y^2 - 48x - 16y - \quad 32 = 0} & \Longrightarrow (y-8)^2 - 48x = 96 \\ 16x^2 \qquad - 48x \qquad - 160 = 0 & \\ 16(x^2 - 3x - 10) = 0 & \\ (x-5)(x+2) = 0 & \\ x = 5 \text{ or } x = -2 & \end{array}$$

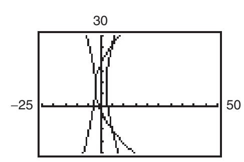
When $x = 5$: $y^2 - 48(5) - 16y - 32 = 0$

$$y^2 - 16y - 272 = 0$$
$$y = 8 \pm 4\sqrt{21}$$

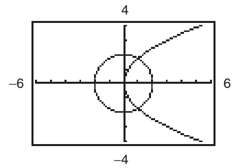
When $x = -2$: $y^2 - 48(-2) - 16y - 32 = 0$

$$y^2 - 16y + 64 = 0$$
$$(y-8)^2 = 0$$
$$y = 8$$

Points of intersection: $\left(5, 8 + 4\sqrt{21}\right), \left(5, 8 - 4\sqrt{21}\right), (-2, 8)$



53.
$$\begin{aligned} x^2 \quad + y^2 - 4 &= 0 \\ 3x - y^2 \quad &= 0 \\ \hline x^2 + 3x \quad - 4 &= 0 \\ (x+4)(x-1) &= 0 \\ x = -4 \quad \text{or} \quad x &= 1 \end{aligned}$$



When $x = -4$: $3(-4) - y^2 = 0$

$$y^2 = -12$$

No real solution

When $x = 1$: $3(1) - y^2 = 0$

$$y^2 = 3$$

$$y = \pm\sqrt{3}$$

The points of intersection are $(1, \sqrt{3})$ and $(1, -\sqrt{3})$.

The standard forms of the equations are:

$x^2 + y^2 = 4$

$y^2 = 3x$

54. $4x^2 + 9y^2 - 36y = 0 \implies 4x^2 + 9(y-2)^2 = 36$

$x^2 + 9y - 27 = 0 \implies y = -\dfrac{x^2}{9} + 3$

$$\begin{aligned} 4(27 - 9y) + 9y^2 - 36y &= 0 \\ 9y^2 - 72y + 108 &= 0 \\ 9(y-6)(y-2) &= 0 \\ y = 6 \ \text{or} \ y &= 2 \end{aligned}$$

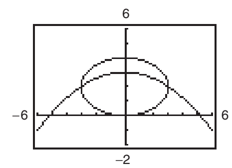
When $y = 6$: $x^2 = 27 - 9(6) = -27$

No real solution

When $y = 2$: $x^2 = 27 - 9(2) = 9$

$$x = \pm 3$$

Points of intersection: $(3, 2), (-3, 2)$



55. $x^2 + 2y^2 - 4x + 6y - 5 = 0$

$-x + y - 4 = 0 \implies y = x + 4$

$$\begin{aligned} x^2 + 2(x+4)^2 - 4x + 6(x+4) - 5 &= 0 \\ x^2 + 2(x^2 + 8x + 16) - 4x + 6x + 24 - 5 &= 0 \\ 3x^2 + 18x + 51 &= 0 \\ 3(x^2 + 6x + 17) &= 0 \\ x^2 + 6x + 17 &= 0 \\ x^2 + 6x + 9 &= -17 + 9 \\ (x+3)^2 &= -8 \end{aligned}$$

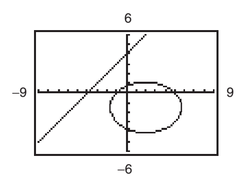
No real solution

No points of intersection

The standard forms of the equations are:

$$\frac{(x-2)^2}{\frac{27}{2}} + \frac{\left(y + \frac{3}{2}\right)^2}{\frac{27}{4}} = 1$$

$x - y = -4$



56. $x^2 + 2y^2 - 4x + 6y - 5 = 0 \implies 2(x-2)^2 + 4\left(y + \frac{3}{2}\right)^2 = 27$

$x^2 - 4x - y + 4 = 0 \implies y = x^2 - 4x + 4$

$$\begin{aligned} y - 4 + 2y^2 + 6y - 5 &= 0 \\ 2y^2 + 7y - 9 &= 0 \\ (2y+9)(y-1) &= 0 \\ y = -\tfrac{9}{2} \ \text{or} \ y &= 1 \end{aligned}$$

When $y = 1$: $x^2 - 4x - 1 + 4 = 0$

$$(x-3)(x-1) = 0$$

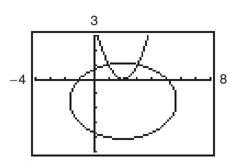
$$x = 1 \ \text{or} \ x = 3$$

When $y = -\frac{9}{2}$: $x^2 - 4x - \left(-\frac{9}{2}\right) + 4 = 0$

$$x^2 - 4x + \tfrac{17}{2} = 0$$

No real solution

Points of intersection: $(1, 1), (3, 1)$



57. $xy + x - 2y + 3 = 0 \implies y = \dfrac{-x-3}{x-2}$

$$x^2 + 4y^2 - 9 = 0$$

$$x^2 + 4\left(\frac{-x-3}{x-2}\right)^2 = 9$$

$$x^2(x-2)^2 + 4(-x-3)^2 = 9(x-2)^2$$

$$x^2(x^2 - 4x + 4) + 4(x^2 + 6x + 9) = 9(x^2 - 4x + 4)$$

$$x^4 - 4x^3 + 4x^2 + 4x^2 + 24x + 36 = 9x^2 - 36x + 36$$

$$x^4 - 4x^3 - x^2 + 60x = 0$$

$$x(x+3)(x^2 - 7x + 20) = 0$$

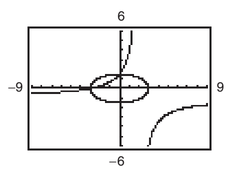
$$x = 0 \text{ or } x = -3$$

Note: $x^2 - 7x + 20 = 0$ has no real solution.

When $x = 0$: $\quad y = \dfrac{-0-3}{0-2} = \dfrac{3}{2}$

When $x = -3$: $y = \dfrac{-(-3)-3}{-3-2} = 0$

The points of intersection are $\left(0, \frac{3}{2}\right), (-3, 0)$.



58. $5x^2 - 2xy + 5y^2 - 12 = 0$

$$x + y - 1 = 0 \implies y = 1 - x$$

$$5x^2 - 2x(1-x) + 5(1-x)^2 - 12 = 0$$

$$5x^2 - 2x + 2x^2 + 5(1 - 2x + x^2) - 12 = 0$$

$$5x^2 - 2x + 2x^2 + 5 - 10x + 5x^2 - 12 = 0$$

$$12x^2 - 12x - 7 = 0$$

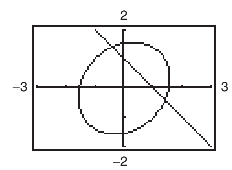
$$x = \frac{3 \pm \sqrt{30}}{6}$$

When $x = \dfrac{3+\sqrt{30}}{6}$: $y = 1 - \dfrac{3+\sqrt{30}}{6} = \dfrac{3-\sqrt{30}}{6}$

When $x = \dfrac{3-\sqrt{30}}{6}$: $y = 1 - \dfrac{3-\sqrt{30}}{6} = \dfrac{3+\sqrt{30}}{6}$

Points of intersection:

$$\left(\frac{1}{6}\left(3+\sqrt{30}\right), \frac{1}{6}\left(3-\sqrt{30}\right)\right), \left(\frac{1}{6}\left(3-\sqrt{30}\right), \frac{1}{6}\left(3+\sqrt{30}\right)\right)$$



59. $x^2 + xy + ky^2 + 6x + 10 = 0$

$B^2 - 4AC = 1^2 - 4(1)(k) = 1 - 4k > 0 \implies -4k > -1 \implies k < \frac{1}{4}$

True. For the graph to be a hyperbola, the discriminant must be greater than zero.

60. False. The coefficients of the new equation, after rotation of axes, are obtained by making the substitutions.

$x = x'\cos\theta - y'\sin\theta$

$y = x'\sin\theta + y'\cos\theta$

61. $r^2 = x^2 + y^2 = (x'\cos\theta - y'\sin\theta)^2 + (y'\cos\theta + x'\sin\theta)^2$

$= (x')^2\cos^2\theta - 2x'y'\cos\theta\sin\theta + (y')^2\sin^2\theta + (y')^2\cos^2\theta + 2x'y'\cos\theta\sin\theta + (x')^2\sin^2\theta$

$= (x')^2(\cos^2\theta + \sin^2\theta) + (y')^2(\sin^2\theta + \cos^2\theta) = (x')^2 + (y')^2$

Thus, $(x')^2 + (y')^2 = r^2$.

62. In Exercise 14, the equation of the rotated ellipse is:

$$\frac{(x')^2}{1} + \frac{(y')^2}{4} = 1$$

$a^2 = 4 \implies a = 2$

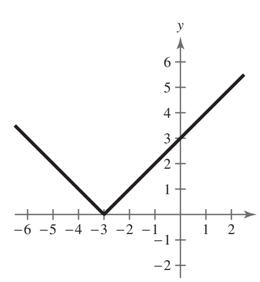
$b^2 = 1 \implies b = 1$

Length of major axis is $2a = 2(2) = 4$.

Length of minor axis is $2b = 2(1) = 2$.

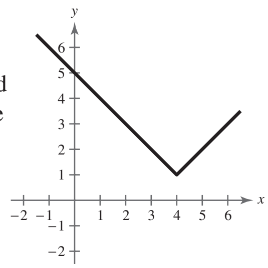
63. $f(x) = |x + 3|$

Shift the graph of $y = |x|$ three units to the left.



64. $f(x) = |x - 4| + 1$

The graph of the function $f(x)$ is the graph of $|x|$ shifted four units to the right and one unit upward.

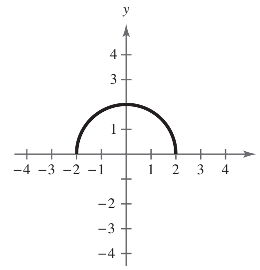


65. $g(x) = \sqrt{4 - x^2}$

$y^2 = 4 - x^2$

$x^2 + y^2 = 4$

$g(x)$ is the top half of this circle since $y \geq 0$.



66. $g(x) = \sqrt{3x - 2}$

x-intercept: $0 = \sqrt{3x - 2}$

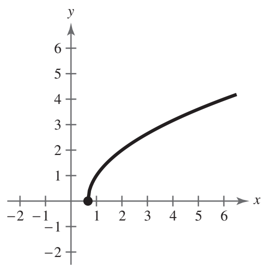
$0^2 = 3x - 2$

$2 = 3x$

$\frac{2}{3} = x, \ \left(\frac{2}{3}, 0\right)$

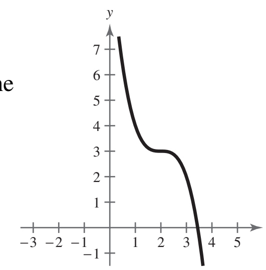
Domain: $\left[\frac{2}{3}, \infty\right)$

x	$\frac{2}{3}$	1	2	$\frac{11}{3}$
y	0	1	2	3



67. $h(t) = -(t - 2)^3 + 3$

Reflect the graph of $y = x^3$ about the x-axis, shift it to the right two units, and upward three units.



68. $h(t) = \frac{1}{2}(t + 4)^3$

y-intercept:

$h(0) = \frac{1}{2}(0 + 4)^3$

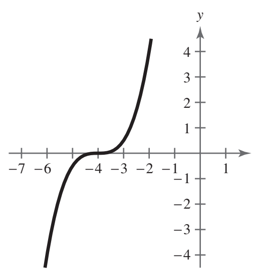
$= 32, \ (0, 32)$

x-intercept: $0 = \frac{1}{2}(t + 4)^3$

$0 = t + 4$

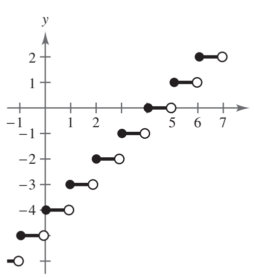
$t = -4, \ (-4, 0)$

t	-6	-4	-2	-1	0	1
$h(t)$	-4	0	4	$\frac{27}{2}$	32	$\frac{125}{2}$

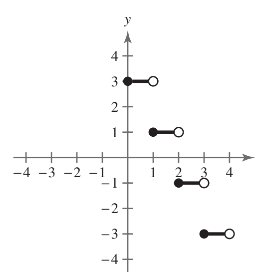


69. $f(t) = [\![t - 5]\!] + 1$

Shift the graph of $y = [\![x]\!]$ five units to the right and upward one unit.



70. $f(t) = -2[\![t]\!] + 3$



71. Area $= \frac{1}{2}ab \sin C$

$= \frac{1}{2}(8)(12) \sin 110°$

≈ 45.11 square units

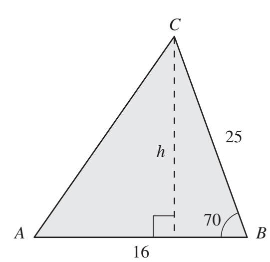
72. $\sin 70° = \dfrac{h}{25}$

$h = 25 \sin 70°$

Area $= \dfrac{1}{2}$(base)(height)

$= \dfrac{1}{2}(16)(25 \sin 70°)$

≈ 187.9



73. $s = \dfrac{a+b+c}{2} = \dfrac{11+18+10}{2} = 19.5$

Area $= \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{(19.5)(8.5)(1.5)(9.5)} \approx 48.60$ square units

74. Law of Cosines:

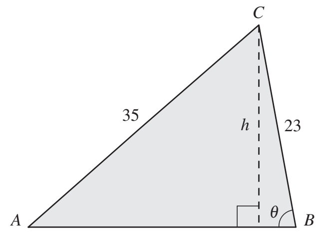
$$35^2 = 23^2 + 27^2 - (2)(23)(27)\cos\theta$$

$$\cos(\theta) \approx \frac{11}{414}$$

$$\theta = \cos^{-1}\left(\frac{11}{414}\right)$$

$$\frac{h}{23} = \sin\theta = \sin\left(\cos^{-1}\left(\frac{11}{414}\right)\right)$$

$$h = 23\sin\left(\cos^{-1}\left(\frac{11}{414}\right)\right)$$



Not drawn to scale

$$\text{Area} = \frac{1}{2}(\text{base})(\text{height}) = \left(\frac{1}{2}\right)(27)\left(23\sin\left(\cos^{-1}\left(\frac{11}{414}\right)\right)\right) \approx 310.4$$

Section 10.6 Parametric Equations

- If f and g are continuous functions of t on an interval I, then the set of ordered pairs $(f(t), g(t))$ is a *plane curve* C. The equations $x = f(t)$ and $y = g(t)$ are *parametric equations* for C and t is the *parameter*.
- To eliminate the parameter:
  (a) Solve for t in one equation and substitute into the second equation.
  (b) Use trigonometric identities.
- You should be able to find the parametric equations for a graph.

Vocabulary Check

1. plane curve; parametric; parameter

2. orientation

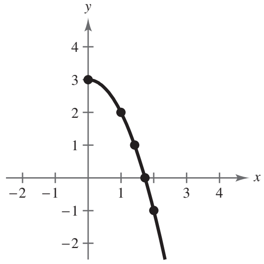
3. eliminating the parameter

1. $x = \sqrt{t}, y = 3 - t$

(a)

t	0	1	2	3	4
x	0	1	$\sqrt{2}$	$\sqrt{3}$	2
y	3	2	1	0	-1

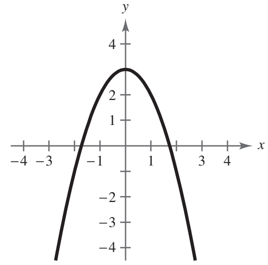
(b)



(c) $x = \sqrt{t} \implies x^2 = t$

$y = 3 - t \implies y = 3 - x^2$

The graph of the parametric equations only shows the right half of the parabola, whereas the rectangular equation yields the entire parabola.

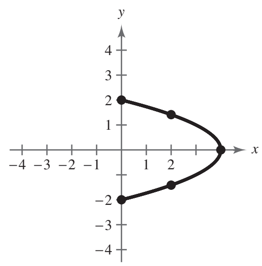


2. $x = 4\cos^2\theta, y = 2\sin\theta$

(a)

θ	$-\pi/2$	$-\pi/4$	0	$\pi/4$	$\pi/2$
x	0	2	4	2	0
y	-2	$-\sqrt{2}$	0	$\sqrt{2}$	2

(b)



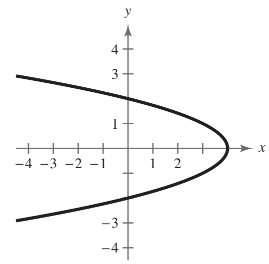
(c) $\dfrac{x}{4} = \cos^2\theta, \dfrac{y}{2} = \sin\theta$

$$\cos^2\theta + \sin^2\theta = 1$$

$$\frac{x}{4} + \left(\frac{y}{2}\right)^2 = 1$$

$$\frac{x}{4} + \frac{y^2}{4} = 1$$

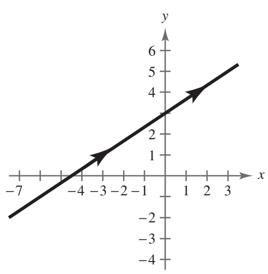
$$x = -y^2 + 4$$



The rectangular version of the graph continues into the second and third quadrants.

3. (a) $x = 3t - 3, y = 2t + 1$

t	-2	-1	0	1	2
x	-9	-6	-3	0	3
y	-3	-1	1	3	5

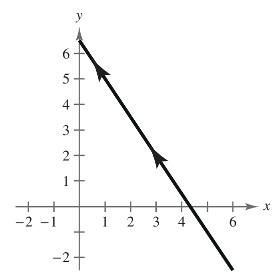


(b) $x = 3t - 3 \implies t = \dfrac{x+3}{3}$

$y = 2t + 1 \implies y = \dfrac{2}{3}(x+3) + 1 = \dfrac{2}{3}x + 3$

4. (a) $x = 3 - 2t, y = 2 + 3t$

t	-3	-2	-1	0	1	2	3
x	9	7	5	3	1	-1	-3
y	-7	-4	-1	2	5	8	11



(b) $x = 3 - 2t \implies t = -\frac{1}{2}x + \frac{3}{2}$

$$y = 2 + 3t$$

$$y = 2 + 3\left(-\tfrac{1}{2}x + \tfrac{3}{2}\right)$$

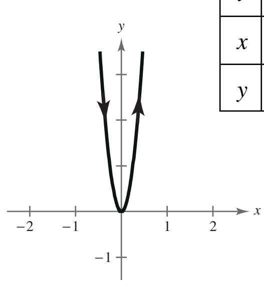
$$y = 2 - \tfrac{3}{2}x + \tfrac{9}{2}$$

$$2y = 4 - 3x + 9$$

$$3x + 2y - 13 = 0$$

5. (a) $x = \frac{1}{4}t, y = t^2$

t	-2	-1	0	1	2
x	$-\frac{1}{2}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$
y	4	1	0	1	4

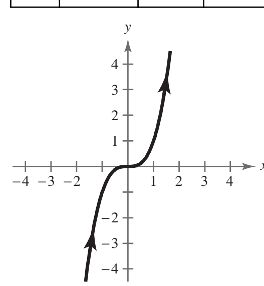


(b) $x = \frac{1}{4}t \implies t = 4x$

$y = t^2 \implies y = 16x^2$

6. (a) $x = t, y = t^3$

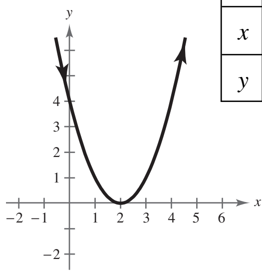
t	-3	-2	-1	0	1	2	3
x	-3	-2	-1	0	1	2	3
y	-27	-8	-1	0	1	8	27



(b) $x = t, y = t^3, y = x^3$

7. (a) $x = t + 2, y = t^2$

t	-2	-1	0	1	2
x	0	1	2	3	4
y	4	1	0	1	4

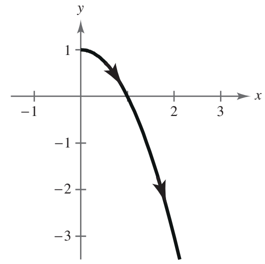


(b) $x = t + 2 \implies t = x - 2$

$y = t^2 \implies y = (x - 2)^2 = x^2 - 4x + 4$

8. (a) $x = \sqrt{t}, y = 1 - t$

t	0	1	2	3
x	0	1	$\sqrt{2}$	$\sqrt{3}$
y	1	0	-1	-2

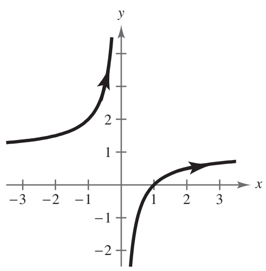


(b) $x = \sqrt{t} \implies x^2 = t,\ t \geq 0$

$y = 1 - t = 1 - x^2,\ x \geq 0$

9. (a) $x = t + 1, y = \dfrac{t}{t + 1}$

t	-3	-2	0	1	2
x	-2	-1	1	2	3
y	$\frac{3}{2}$	2	0	$\frac{1}{2}$	$\frac{2}{3}$

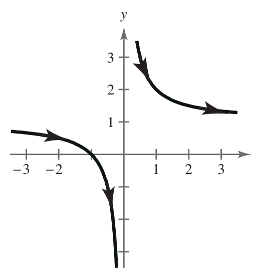


(b) $x = t + 1 \implies t = x - 1$

$y = \dfrac{t}{t + 1} \implies y = \dfrac{x - 1}{x}$

10. (a) $x = t - 1, y = \dfrac{t}{t - 1}$

t	-3	-2	-1	0	2	3
x	-4	-3	-2	-1	1	2
y	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{1}{2}$	0	2	$\frac{3}{2}$

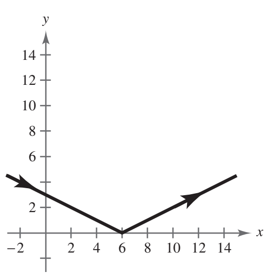


(b) $x = t - 1 \implies t = x + 1$

$y = \dfrac{t}{t - 1} = \dfrac{x + 1}{x + 1 - 1} = \dfrac{x + 1}{x}$

11. (a) $x = 2(t + 1), y = |t - 2|$

t	0	2	4	6	8	10
x	2	6	10	14	18	22
y	2	0	2	4	6	8

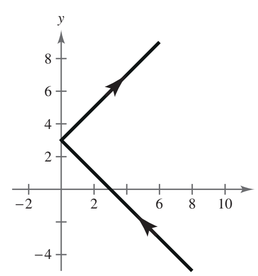


(b) $x = 2(t + 1) \implies \dfrac{x}{2} - 1 = t$ or $t = \dfrac{x - 2}{2}$

$y = |t - 2| \implies y = \left|\dfrac{x}{2} - 1 - 2\right| = \left|\dfrac{x}{2} - 3\right|$

12. (a) $x = |t - 1|, y = t + 2$

t	-3	-2	-1	0	1	2	3
x	4	3	2	1	0	1	2
y	-1	0	1	2	3	4	5



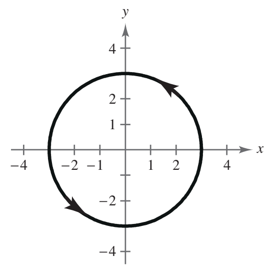
(b) $x = |t - 1|$

$y = t + 2 \implies t = y - 2 \implies x = |y - 3|$

OR $y = x + 3, x \geq 0$ and $y = -x + 3, x \geq 0$

13. (a) $x = 3\cos\theta, y = 3\sin\theta$

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
x	3	0	-3	0	3
y	0	3	0	-3	0



(b) $x = 3\cos\theta \Rightarrow \left(\frac{x}{3}\right)^2 = \cos^2\theta$

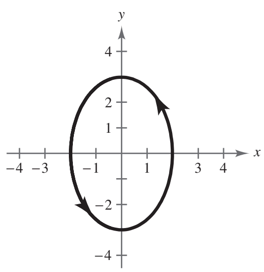
$y = 3\sin\theta \Rightarrow \left(\frac{y}{3}\right)^2 = \sin^2\theta$

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$

$$\frac{x^2}{9} + \frac{y^2}{9} = 1$$

14. (a) $x = 2\cos\theta, y = 3\sin\theta$

θ	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$	2π
x	2	$\sqrt{2}$	0	$-\sqrt{2}$	-2	$-\sqrt{2}$	0	$\sqrt{2}$	2
y	0	$3\sqrt{2}/2$	3	$3\sqrt{2}/2$	0	$-3\sqrt{2}/2$	-3	$-3\sqrt{2}/2$	0



(b) $x = 2\cos\theta$

$y = 3\sin\theta$

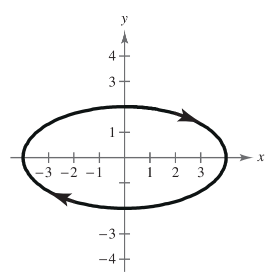
$\cos^2\theta + \sin^2\theta = 1$

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

15. (a) $x = 4\sin 2\theta, y = 2\cos 2\theta$

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
x	0	4	0	-4	0
y	2	0	-2	0	2



(b) $x = 4\sin 2\theta \Rightarrow \left(\frac{x}{4}\right)^2 = \sin^2 2\theta$

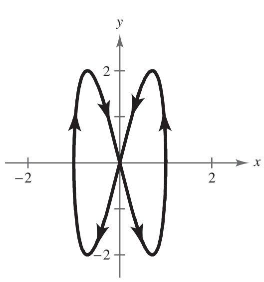
$y = 2\cos 2\theta \Rightarrow \left(\frac{y}{2}\right)^2 = \cos^2 2\theta$

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$$

$$\frac{x^2}{16} + \frac{y^2}{4} = 1$$

16. (a) $x = \cos\theta, y = 2\sin 2\theta$

θ	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$	2π
x	1	$\sqrt{2}/2$	0	$-\sqrt{2}/2$	-1	$-\sqrt{2}/2$	0	$\sqrt{2}/2$	1
y	0	2	0	-2	0	2	0	-2	0



(b) $x = \cos\theta$

$y = 2\sin 2\theta$

$y = 2\sin 2\theta$

$y = 2(2\sin\theta\cos\theta)$

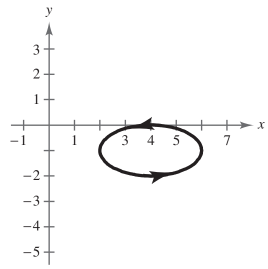
$y^2 = 16\sin^2\theta\cos^2\theta$

$y^2 = 16(1 - x^2)x^2$

$y^2 = 16x^2(1 - x^2)$

17. (a) $x = 4 + 2\cos\theta, y = -1 + \sin\theta$

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
x	6	4	2	4	6
y	-1	0	-1	-2	-1



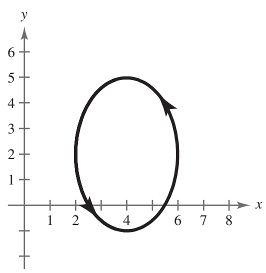
(b) $x = 4 + 2\cos\theta \Rightarrow \left(\frac{x-4}{2}\right)^2 = \cos^2\theta$

$y = -1 + \sin\theta \Rightarrow (y+1)^2 = \sin^2\theta$

$$\frac{(x-4)^2}{4} + \frac{(y+1)^2}{1} = 1$$

18. (a) $x = 4 + 2\cos\theta, y = 2 + 3\sin\theta$

θ	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$	2π
x	6	$4 + \sqrt{2}$	4	$4 - \sqrt{2}$	2	$4 - \sqrt{2}$	4	$4 + \sqrt{2}$	6
y	2	$2 + (3\sqrt{2}/2)$	5	$2 + (3\sqrt{2}/2)$	2	$2 - (3\sqrt{2}/2)$	-1	$2 - (3\sqrt{2}/2)$	2



(b) $x = 4 + 2\cos\theta$

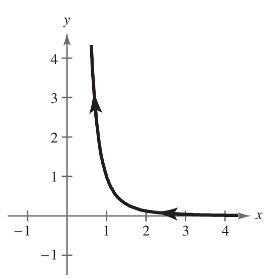
$y = 2 + 3\sin\theta$

$\cos^2\theta + \sin^2\theta = 1$

$$\frac{(x-4)^2}{4} + \frac{(y-2)^2}{9} = 1$$

19. (a) $x = e^{-t}, y = e^{3t}$

t	-2	-1	0	1	2
x	7.3891	2.7183	1	0.3679	0.1353
y	0.0025	0.0498	1	20.0855	403.4288



(b) $x = e^{-t} \Rightarrow \frac{1}{x} = e^t$

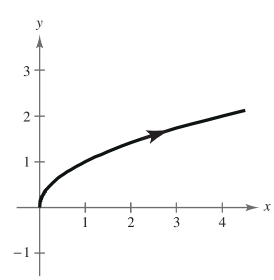
$y = e^{3t} \Rightarrow y = (e^t)^3$

$y = \left(\frac{1}{x}\right)^3$

$y = \frac{1}{x^3}, \ x > 0, y > 0$

20. (a) $x = e^{2t}, y = e^t$

t	-3	-2	-1	0	1	2
x	0.0025	0.0183	0.1353	1	7.3891	54.5982
y	0.0498	0.1353	0.3679	1	2.7183	7.3891



(b) $x = e^{2t}$

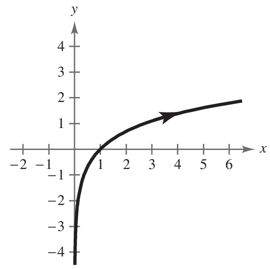
$y = e^t \Rightarrow y^2 = e^{2t}$

$x = e^{2t} = y^2$

$y^2 = x, y > 0$

21. (a) $x = t^3, y = 3 \ln t$

t	$\frac{1}{2}$	1	2	3	4
x	$\frac{1}{8}$	1	8	27	64
y	-2.0794	0	2.0794	3.2958	4.1589



(b) $x = t^3 \implies x^{1/3} = t$

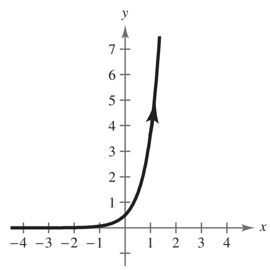
$y = 3 \ln t \implies y = \ln t^3$

$y = \ln(x^{1/3})^3$

$y = \ln x$

22. (a) $x = \ln 2t, y = 2t^2$

t	1	2	3	4
x	0.6931	1.3863	1.7918	2.0794
y	2	8	18	32



(b) $x = \ln 2t \implies t = \frac{1}{2}e^x$

$y = 2t^2$

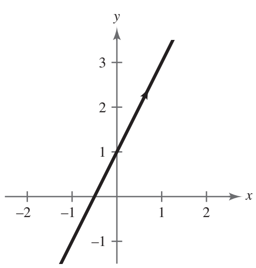
$y = 2t^2 = 2\left(\frac{1}{2}e^x\right)^2 = \frac{1}{2}e^{2x}$

23. By eliminating the parameter, each curve becomes $y = 2x + 1$.

(a) $x = t$
$y = 2t + 1$
There are no restrictions on x and y.

Domain: $(-\infty, \infty)$

Orientation: Left to right



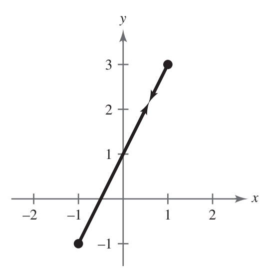
(b) $x = \cos\theta \implies -1 \le x \le 1$

$y = 2\cos\theta + 1 \implies -1 \le y \le 3$

The graph oscillates.

Domain: $[-1, 1]$

Orientation: Depends on θ

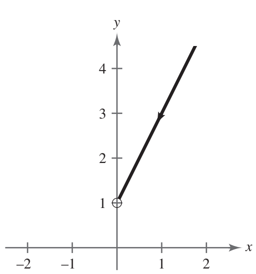


(c) $x = e^{-t} \implies x > 0$

$y = 2e^{-t} + 1 \implies y > 1$

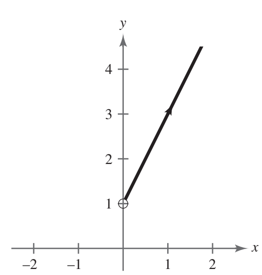
Domain: $(0, \infty)$

Orientation: Downward or right to left



(d) $x = e^t \implies x > 0$
$y = 2e^t + 1 \implies y > 1$
Domain: $(0, \infty)$

Orientation: Upward or left to right

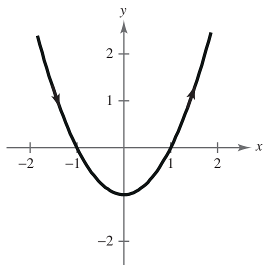


24. By eliminating the parameter, each curve represents a portion of $y = x^2 - 1$.

(a) $x = t$

$y = t^2 - 1$

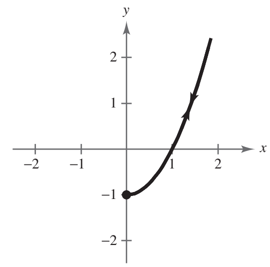
There are no restrictions on x.



Domain: $(-\infty, \infty)$
Orientation: Left to right

(b) $x = t^2 \implies x \ge 0$

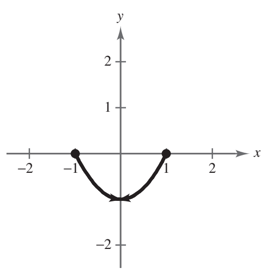
$y = t^4 - 1$



Domain: $[0, \infty)$
Orientation: Depends on t

(c) $x = \sin t \implies -1 \le x \le 1$

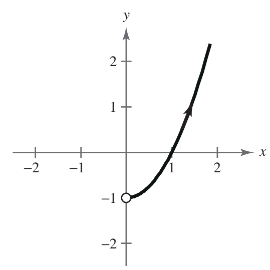
$y = \sin^2 t - 1$



Domain: $[-1, 1]$
Orientation: Depends on t

(d) $x = e^t \implies x > 0$

$y = e^{2t} - 1$



Domain: $(0, \infty)$
Orientation: Left to right

25. $x = x_1 + t(x_2 - x_1),\ y = y_1 + t(y_2 - y_1)$

$$\frac{x - x_1}{x_2 - x_1} = t$$

$$y = y_1 + \left(\frac{x - x_1}{x_2 - x_1}\right)(y_2 - y_1)$$

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) = m(x - x_1)$$

26. $x = h + r\cos\theta,\ y = k + r\sin\theta$

$$\cos\theta = \frac{x - h}{r},\ \sin\theta = \frac{y - k}{r}$$

$$\cos^2\theta + \sin^2\theta = \frac{(x - h)^2}{r^2} + \frac{(y - k)^2}{r^2} = 1$$

$$(x - h)^2 + (y - k)^2 = r^2$$

27. $x = h + a\cos\theta,\ y = k + b\sin\theta$

$$\frac{x - h}{a} = \cos\theta,\ \frac{y - k}{b} = \sin\theta$$

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$$

28. $x = h + a\sec\theta,\ y = k + b\tan\theta$

$$\frac{x - h}{a} = \sec\theta,\ \frac{y - k}{b} = \tan\theta$$

$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$$

29. From Exercise 25 we have:

$$x = 0 + t(6 - 0) = 6t$$

$$y = 0 + t(-3 - 0) = -3t$$

30. Line through $(2, 3)$ and $(6, -3)$

From Exercise 25 we have:

$$x = x_1 + t(x_2 - x_1) = 2 + t(6 - 2) = 2 + 4t$$

$$y = y_1 + t(y_2 - y_1) = 3 + t(-3 - 3) = 3 - 6t$$

31. From Exercise 26 we have:

$x = 3 + 4\cos\theta$

$y = 2 + 4\sin\theta$

32. Circle with center $(-3, 2)$; radius: 5

From Exercise 26 we have:

$x = h + r\cos\theta = -3 + 5\cos\theta$

$y = k + r\sin\theta = 2 + 5\sin\theta$

33. Vertices: $(\pm 4, 0) \implies (h, k) = (0, 0)$ and $a = 4$

Foci: $(\pm 3, 0) \implies c = 3$

$c^2 = a^2 - b^2 \implies 9 = 16 - b^2 \implies b = \sqrt{7}$

From Exercise 27 we have:

$x = 4\cos\theta$

$y = \sqrt{7}\sin\theta$

34. Ellipse

Vertices: $(4, 7), (4, -3) \implies (h, k) = (4, 2), a = 5$

Foci: $(4, 5), (4, -1) \implies c = 3$

$b^2 = a^2 - c^2 = 25 - 9 = 16 \implies b = 4$

From Exercise 27 we have:

$x = h + b\cos\theta = 4 + 4\cos\theta$

$y = k + a\sin\theta = 2 + 5\sin\theta$

35. Vertices: $(\pm 4, 0) \implies (h, k) = (0, 0)$ and $a = 4$

Foci: $(\pm 5, 0) \implies c = 5$

$c^2 = a^2 + b^2 \implies 25 = 16 + b^2 \implies b = 3$

From Exercise 28 we have:

$x = 4\sec\theta$

$y = 3\tan\theta$

36. Hyperbola

Vertices: $(\pm 2, 0) \implies (h, k) = (0, 0), a = 2$

Foci: $(\pm 4, 0) \implies c = 4$

$b^2 = c^2 - a^2 = 16 - 4 \implies b = 2\sqrt{3}$

From Exercise 28 we have:

$x = h + a\sec\theta = 2\sec\theta$

$y = k + b\tan\theta = 2\sqrt{3}\tan\theta$

37. $y = 3x - 2$

(a) $t = x \implies x = t$ and $y = 3t - 2$

(b) $t = 2 - x \implies x = -t + 2$ and

$y = 3(-t + 2) - 2 = -3t + 4$

38. $x = 3y - 2$

(a) $t = x, x = t, y = \frac{1}{3}(t + 2)$

(b) $t = 2 - x, x = 2 - t, y = \frac{1}{3}(x + 2) = \frac{1}{3}(4 - t)$

39. $y = x^2$

(a) $t = x \implies x = t$ and $y = t^2$

(b) $t = 2 - x \implies x = -t + 2$ and

$y = (-t + 2)^2 = t^2 - 4t + 4$

40. $y = x^3$

(a) $t = x, x = t, y = t^3$

(b) $t = 2 - x, x = 2 - t, y = (2 - t)^3$

41. $y = x^2 + 1$

(a) $t = x \implies x = t$ and $y = t^2 + 1$

(b) $t = 2 - x \implies x = -t + 2$ and

$y = (-t + 2)^2 + 1 = t^2 - 4t + 5$

42. $y = 2 - x$

(a) $t = x, x = t, y = 2 - t$

(b) $t = 2 - x, x = 2 - t, y = 2 - (2 - t) = t$

43. $y = \dfrac{1}{x}$

(a) $t = x \implies x = t$ and $y = \dfrac{1}{t}$

(b) $t = 2 - x \implies x = -t + 2$ and $y = \dfrac{1}{-t + 2} = \dfrac{-1}{t - 2}$

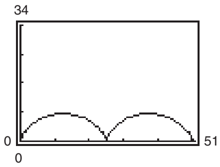
44. $y = \dfrac{1}{2x}$

(a) $t = x, x = t, y = \dfrac{1}{2t}$

(b) $t = 2 - x, x = 2 - t, y = \dfrac{1}{2(2 - t)} = \dfrac{1}{4 - 2t}$

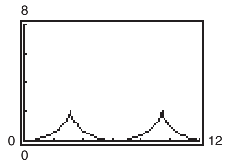
45. $x = 4(\theta - \sin\theta)$

$y = 4(1 - \cos\theta)$



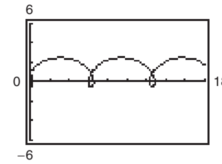
46. $x = \theta + \sin\theta$

$y = 1 - \cos\theta$



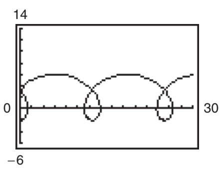
47. $x = \theta - \frac{3}{2}\sin\theta$

$y = 1 - \frac{3}{2}\cos\theta$



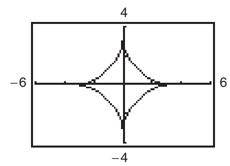
48. $x = 2\theta - 4\sin\theta$

$y = 2 - 4\cos\theta$



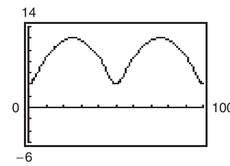
49. $x = 3\cos^3\theta$

$y = 3\sin^3\theta$



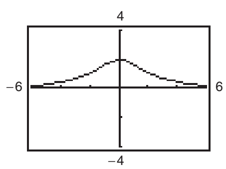
50. $x = 8\theta - 4\sin\theta$

$y = 8 - 4\cos\theta$



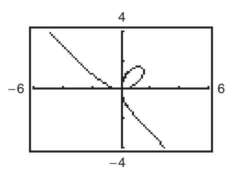
51. $x = 2\cot\theta$

$y = 2\sin^2\theta$



52. $x = \dfrac{3t}{1 + t^3}$

$y = \dfrac{3t^2}{1 + t^3}$



53. $x = 2\cos\theta \implies -2 \le x \le 2$

$y = \sin 2\theta \implies -1 \le y \le 1$

Matches graph (b).

Domain: $[-2, 2]$

Range: $[-1, 1]$

54. $x = 4\cos^3\theta \implies -4 \le x \le 4$

$y = 6\sin^3\theta \implies -6 \le y \le 6$

Matches graph (c).

Domain: $[-4, 4]$

Range: $[-6, 6]$

55. $x = \frac{1}{2}(\cos\theta + \theta\sin\theta)$

$y = \frac{1}{2}(\sin\theta - \theta\cos\theta)$

Matches graph (d).

Domain: $(-\infty, \infty)$

Range: $(-\infty, \infty)$

56. $x = \frac{1}{2}\cot\theta \implies -\infty < x < \infty$

$y = 4\sin\theta\cos\theta \implies -2 \le y \le 2$

Matches graph (a).

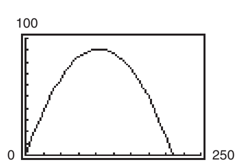
Domain: $(-\infty, \infty)$

Range: $[-2, 2]$

57. $x = (v_0\cos\theta)t$ and $y = h + (v_0\sin\theta)t - 16t^2$

(a) $\theta = 60°$, $v_0 = 88$ ft/sec

$x = (88\cos 60°)t$ and $y = (88\sin 60°)t - 16t^2$

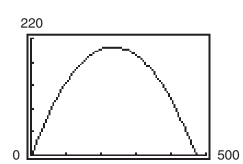


Maximum height: 90.7 feet

Range: 209.6 feet

(b) $\theta = 60°$, $v_0 = 132$ ft/sec

$x = (132\cos 60°)t$ and $y = (132\sin 60°)t - 16t^2$



Maximum height: 204.2 feet

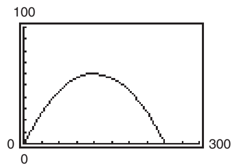
Range: 471.6 feet

—CONTINUED—

57. —CONTINUED—

(c) $\theta = 45°$, $v_0 = 88$ ft/sec

$x = (88 \cos 45°)t$ and $y = (88 \sin 45°)t - 16t^2$

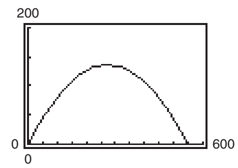


Maximum height: 60.5 ft

Range: 242.0 ft

(d) $\theta = 45°$, $v_0 = 132$ ft/sec

$x = (132 \cos 45°)t$ and $y = (132 \sin 45°)t - 16t^2$



Maximum height: 136.1 ft

Range: 544.5 ft

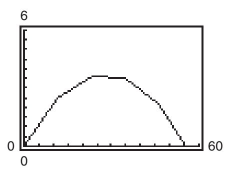
58. $x = (v_0 \cos \theta)t$

$y = h + (v_0 \sin \theta)t - 16t^2$

(a) $\theta = 15°$, $v_0 = 60$ ft/sec

Maximum height: 3.8 feet

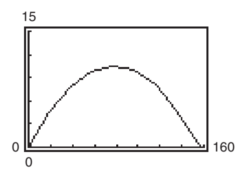
Range: 56.3 feet



(b) $\theta = 15°$, $v_0 = 100$ ft/sec

Maximum height: 10.5 feet

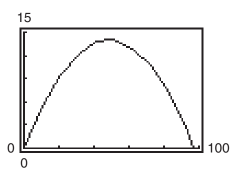
Range: 156.3 feet



(c) $\theta = 30°$, $v_0 = 60$ ft/sec

Maximum height: 14.1 feet

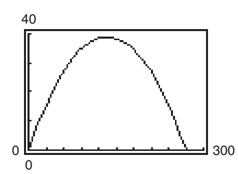
Range: 97.4 feet



(d) $\theta = 30°$, $v_0 = 100$ ft/sec

Maximum height: 39.1 feet

Range: 270.6 feet



59. (a) 100 miles per hour $= 100\left(\frac{5280}{3600}\right)$ ft/sec $= \frac{440}{3}$ ft/sec

$x = \left(\frac{440}{3} \cos \theta\right)t \approx (146.67 \cos \theta)t$

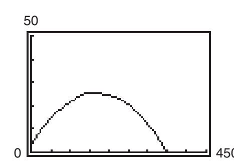
$y = 3 + \left(\frac{440}{3} \sin \theta\right)t - 16t^2 \approx 3 + (146.67 \sin \theta)t - 16t^2$

(b) For $\theta = 15°$, we have:

$x = \left(\frac{440}{3} \cos 15°\right)t \approx 141.7t$

$y = 3 + \left(\frac{440}{3} \sin 15°\right)t - 16t^2 \approx 3 + 38.0t - 16t^2$

The ball hits the ground inside the ballpark, so it is not a home run.

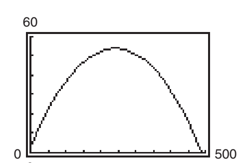


(c) For $\theta = 23°$, we have:

$x = \left(\frac{440}{3} \cos 23°\right)t \approx 135.0t$

$y = 3 + \left(\frac{440}{3} \sin 23°\right)t - 16t^2 \approx 3 + 57.3t - 16t^2$

The ball easily clears the 7-foot fence at 408 feet so it is a home run.



(d) Find θ so that $y = 7$ when $x = 408$ by graphing the parametric equations for θ values between 15° and 23°. This occurs when $\theta \approx 19.3°$.

60. (a) $x = (v_0 \cos \theta)t$

$y = h + (v_0 \sin \theta)t - 16t^2$

$h = 5, v_0 = 240, \theta = 10°$

$x = (240 \cos 10°)t$

$y = 5 + (240 \sin 10°)t - 16t^2$

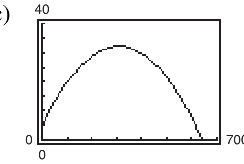
(b) $y = 5 + (240 \sin 10°)t - 16t^2 = 0$

$$t = \frac{-240 \sin 10° \pm \sqrt{(240 \sin 10°)^2 - 4(-16)5}}{2(-16)}$$

$t \approx -0.1149, 2.7196$

Distance traveled before arrow hits ground:

$(240 \cos 10°)(2.7196) \approx 643$ feet

(c) 

Maximum height: 32.1 feet

(d) Time arrow is in the air: approximately 2.72 seconds (see part b)

61. $x = (v_0 \cos \theta)t \implies t = \dfrac{x}{v_0 \cos \theta}$

$$\begin{aligned} y &= h + (v_0 \sin \theta)t - 16t^2 \\ &= h + (v_0 \sin \theta)\left(\frac{x}{v_0 \cos \theta}\right) - 16\left(\frac{x}{v_0 \cos \theta}\right)^2 \\ &= h + (\tan \theta)x - \frac{16x^2}{v_0^2 \cos^2 \theta} \\ &= -\frac{16 \sec^2 \theta}{v_0^2}x^2 + (\tan \theta)x + h \end{aligned}$$

62. $y = 7 + x - 0.02x^2$

(a) Exercise 61 result: $y = -\dfrac{16 \sec^2 \theta}{v_0^2}x^2 + (\tan \theta)x + h$

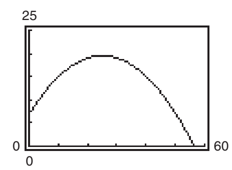
$h = 7$

$\tan \theta = 1 \implies \theta = 45°$

$\dfrac{16 \sec^2 45°}{v_0^2} = 0.02 \implies v_0 = 40$

$x = (v_0 \cos \theta)t = (40 \cos 45°)t$

$y = h + (v_0 \sin \theta)t - 16t^2 \approx 7 + (40 \sin 45°)t - 16t^2$

(b) 

(c) Maximum height: 19.5 feet

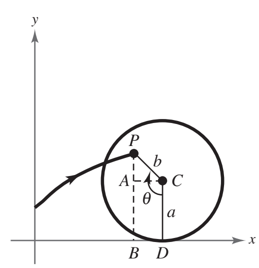
Range: 56.2 feet

63. When the circle has rolled θ radians, the center is at $(a\theta, a)$.

$$\begin{aligned} \sin \theta &= \sin(180° - \theta) \\ &= \frac{|AC|}{b} = \frac{|BD|}{b} \implies |BD| = b \sin \theta \end{aligned}$$

$$\begin{aligned} \cos \theta &= -\cos(180° - \theta) \\ &= \frac{|AP|}{-b} \implies |AP| = -b \cos \theta \end{aligned}$$

Therefore, $x = a\theta - b \sin \theta$ and $y = a - b \cos \theta$.



64. The coordinates of point (x, y) can be thought of as the sum of two vectors:

From origin to center of small circle: $\langle 3 \cos \theta, 3 \sin \theta \rangle$

From center of small circle to point (x, y): $\langle \cos \beta, \sin \beta \rangle$

Because the small circle rotates by 2θ when its center has rotated by θ, we have $\beta = \pi + 3\theta$.

$x = 3 \cos \theta + \cos(\pi + 3\theta) = 3 \cos \theta - \cos 3\theta$

$y = 3 \sin \theta + \sin(\pi + 3\theta) = 3 \sin \theta - \sin 3\theta$

65. True

$x = t$

$y = t^2 + 1 \implies y = x^2 + 1$

$x = 3t$

$y = 9t^2 + 1 \implies y = x^2 + 1$

66. False. Since $t^2 \geq 0$, the graph is that of $y = x$ for $x \geq 0$.

67. The use of parametric equations is useful when graphing two functions simultaneously on the same coordinate system. For example, this is useful when tracking the path of an object so the position and the time associated with that position can be determined.

68. For selected values of t, prepare a table of values for $x(t)$ and $y(t)$. Plot the points $(x(t), y(t))$ in the table. Sketch a curve through the points in order of increasing t (this is the *orientation* of the curve).

69.
$$\begin{array}{rcl} 5x - 7y = 11 & \Rightarrow & 5x - 7y = 11 \\ -3x + y = -13 & \Rightarrow & \underline{-21x + 7y = -91} \\ & & -16x \quad = -80 \\ & & x = 5 \end{array}$$

$$5(5) - 7y = 11 \Rightarrow y = 2$$

Solution: $(5, 2)$

70.
$$\begin{cases} 3x + 5y = 9 \Rightarrow 6x + 10y = 18 \\ 4x - 2y = -14 \Rightarrow 20x - 10y = -70 \end{cases}$$

$$26x = -52$$

$$x = -2$$

$$3(-2) + 5y = 9 \Rightarrow y = 3$$

Solution: $(-2, 3)$

71.
$$\begin{array}{rcl} 3a - 2b + c = 8 & \Rightarrow & 9a - 6b + 3c = 24 \\ 2a + b - 3c = -3 & \Rightarrow & \underline{2a + b - 3c = -3} \\ & & 11a - 5b \quad = 21 \end{array}$$

$$\begin{array}{rcl} 2a + b - 3c = -3 & \Rightarrow & 6a + 3b - 9c = -9 \\ a - 3b + 9c = 16 & \Rightarrow & \underline{a - 3b + 9c = 16} \\ & & 7a \quad = 7 \\ & & a \quad = 1 \end{array}$$

$$11(1) - 5b = 21 \Rightarrow b = -2$$

$$3(1) - 2(-2) + c = 8 \Rightarrow c = 1$$

Solution: $(1, -2, 1)$

72.
$$\begin{cases} 5u + 7v + 9w = 4 & \text{Equation 1} \\ u - 2v - 3w = 7 & \text{Equation 2} \\ 8u - 2v + w = 20 & \text{Equation 3} \end{cases}$$

$$\begin{cases} u - 2v - 3w = 7 & \text{Interchange Eq.1 and Eq.2} \\ 5u + 7v + 9w = 4 \\ 8u - 2v + w = 20 \end{cases}$$

$$\begin{cases} u - 2v - 3w = 7 \\ 17v + 24w = -31 & (-5)\text{Eq.1} + \text{Eq.2} \\ 14v + 25w = -36 & (-8)\text{Eq.1} + \text{Eq.3} \end{cases}$$

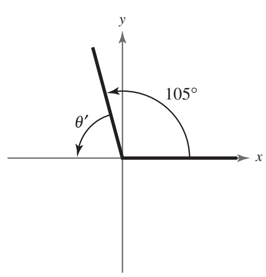
$$\begin{cases} u - 2v - 3w = 7 \\ 17v + 24w = -31 \\ 89w = -178 & (-14)\text{Eq.2} + (17)\text{Eq.3} \end{cases}$$

$$\begin{array}{rcl} 89w = -178 & \Rightarrow & w = -2 \\ 17v + 24(-2) = -31 & \Rightarrow & v = 1 \\ u - 2(1) - 3(-2) = 7 & \Rightarrow & u = 3 \end{array}$$

Solution: $(3, 1, -2)$

73. $\theta = 105°$

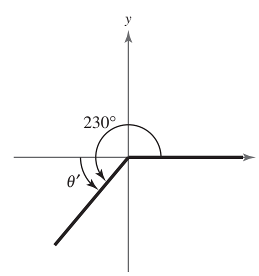
$\theta' = 180° - 105° = 75°$



74. $\theta = 230°$

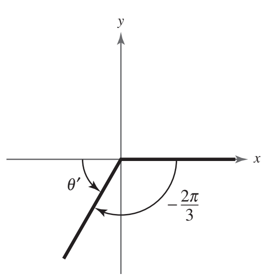
$\theta' = \theta - 180°$

$= 230° - 180° = 50°$



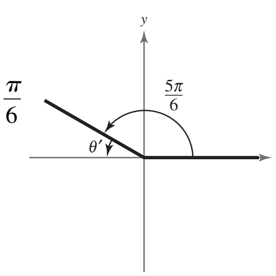
75. $\theta = -\dfrac{2\pi}{3}$

$\theta' = -\dfrac{2\pi}{3} + \pi = \dfrac{\pi}{3}$



76. $\theta = \dfrac{5\pi}{6}$

$\theta' = \pi - \theta = \pi - \dfrac{5\pi}{6} = \dfrac{\pi}{6}$



Section 10.7 Polar Coordinates

- In polar coordinates you do not have unique representation of points. The point (r, θ) can be represented by $(r, \theta \pm 2n\pi)$ or by $(-r, \theta \pm (2n + 1)\pi)$ where n is any integer. The pole is represented by $(0, \theta)$ where θ is any angle.
- To convert from polar coordinates to rectangular coordinates, use the following relationships.
  $x = r\cos\theta$
  $y = r\sin\theta$
- To convert from rectangular coordinates to polar coordinates, use the following relationships.
  $r = \pm\sqrt{x^2 + y^2}$
  $\tan\theta = y/x$
  If θ is in the same quadrant as the point (x, y), then r is positive. If θ is in the opposite quadrant as the point (x, y), then r is negative.
- You should be able to convert rectangular equations to polar form and vice versa.

Vocabulary Check

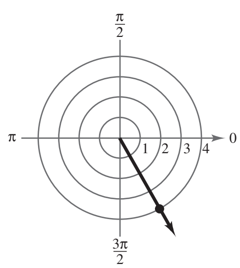
1. pole
2. directed distance; directed angle
3. polar
4. $x = r\cos\theta, \quad \tan\theta = \dfrac{y}{x}$
   $y = r\sin\theta, \quad r^2 = x^2 + y^2$

1. Polar coordinates: $\left(4, -\dfrac{\pi}{3}\right)$

Additional representations:

$$\left(4, -\frac{\pi}{3} + 2\pi\right) = \left(4, \frac{5\pi}{3}\right)$$

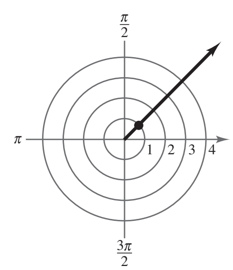
$$\left(-4, -\frac{\pi}{3} - \pi\right) = \left(-4, -\frac{4\pi}{3}\right)$$



2. $\left(-1, -\dfrac{3\pi}{4}\right)$

$\left(1, \dfrac{\pi}{4}\right)$

$\left(-1, \dfrac{5\pi}{4}\right)$

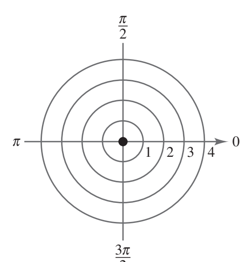


3. Polar coordinates: $\left(0, -\dfrac{7\pi}{6}\right)$

Additional representations:

$$\left(0, -\frac{7\pi}{6} + 2\pi\right) = \left(0, \frac{5\pi}{6}\right)$$

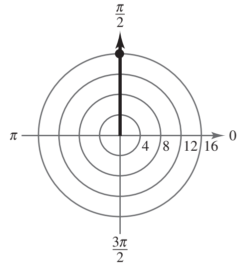
$$\left(0, -\frac{7\pi}{6} + \pi\right) = \left(0, -\frac{\pi}{6}\right) \text{ or } (0, \theta) \text{ for any } \theta,\ -2\pi < \theta < 2\pi$$



4. $\left(16, \frac{5\pi}{2}\right)$

$\left(16, \frac{\pi}{2}\right)$

$\left(-16, \frac{3\pi}{2}\right)$

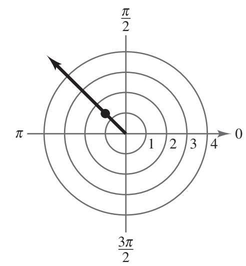


5. Polar coordinates: $(\sqrt{2}, 2.36)$

Additional representations:

$(\sqrt{2}, 2.36 - 2\pi) \approx (\sqrt{2}, -3.92)$

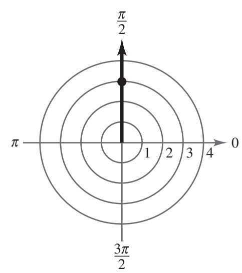
$(-\sqrt{2}, 2.36 - \pi) \approx (-\sqrt{2}, -0.78)$



6. $(-3, -1.57)$

$(3, 1.5716)$

$(-3, 4.7132)$

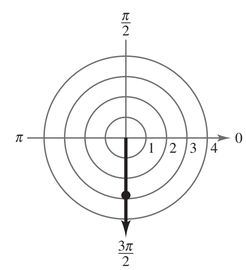


7. Polar coordinates: $(2\sqrt{2}, 4.71)$

Additional representations:

$(2\sqrt{2}, 4.71 - 2\pi) \approx (2\sqrt{2}, -1.57)$

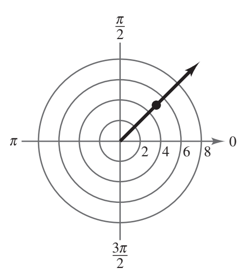
$(-2\sqrt{2}, 2\pi - 4.71) \approx (-2\sqrt{2}, 1.57)$



8. $(-5, -2.36)$

$(5, 0.7816)$

$(-5, 3.9232)$



9. Polar coordinates: $\left(3, \frac{\pi}{2}\right)$

$x = 3 \cos \frac{\pi}{2} = 0$

$y = 3 \sin \frac{\pi}{2} = 3$

Rectangular coordinates: $(0, 3)$

10. Polar coordinates: $\left(3, \frac{3\pi}{2}\right) = (r, \theta)$

$x = r \cos \theta = 3 \cos \frac{3\pi}{2} = 0$

$y = r \sin \theta = 3 \sin \frac{3\pi}{2} = -3$

Rectangular coordinates: $(0, -3)$

11. Polar coordinates: $\left(-1, \frac{5\pi}{4}\right)$

$x = -1 \cos\left(\frac{5\pi}{4}\right) = \frac{\sqrt{2}}{2}, y = -1 \sin\left(\frac{5\pi}{4}\right) = \frac{\sqrt{2}}{2}$

Rectangular coordinates: $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

12. Polar coordinates: $(0, -\pi) = (r, \theta)$

$x = r\cos\theta = 0$

$y = r\sin\theta = 0$

Rectangular coordinates: $(0, 0)$

13. Polar coordinates: $\left(2, \frac{3\pi}{4}\right)$

$x = 2\cos\frac{3\pi}{4} = -\sqrt{2}$

$y = 2\sin\frac{3\pi}{4} = \sqrt{2}$

Rectangular coordinates: $\left(-\sqrt{2}, \sqrt{2}\right)$

14. Polar coordinates: $\left(-2, \frac{7\pi}{6}\right) = (r, \theta)$

$x = r\cos\theta = -2\cos\frac{7\pi}{6} = \sqrt{3}$

$y = r\sin\theta = -2\sin\frac{7\pi}{6} = 1$

Rectangular coordinates: $\left(\sqrt{3}, 1\right)$

15. Polar coordinates: $(-2.5, 1.1)$

$x = -2.5\cos 1.1 \approx -1.1340$

$y = -2.5\sin 1.1 \approx -2.2280$

Rectangular coordinates: $(-1.1340, -2.2280)$

16. Polar coordinates: $(8.25, 3.5) = (r, \theta)$

$x = r\cos\theta = 8.25\cos 3.5 \approx -7.7258$

$y = r\sin\theta = 8.25\sin 3.5 \approx -2.8940$

Rectangular coordinates: $(-7.7258, -2.8940)$

17. Rectangular coordinates: $(1, 1)$

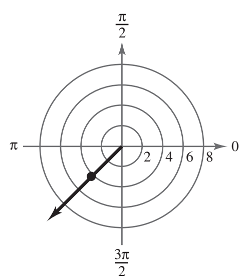
$r = \pm\sqrt{2}$, $\tan\theta = 1$, $\theta = \frac{\pi}{4}$ or $\frac{5\pi}{4}$

Polar coordinates: $\left(\sqrt{2}, \frac{\pi}{4}\right), \left(-\sqrt{2}, \frac{5\pi}{4}\right)$

18. Rectangular coordinates: $(-3, -3)$

$r = 3\sqrt{2}$, $\tan\theta = 1$, $\theta = \frac{5\pi}{4}$

Polar coordinates: $\left(3\sqrt{2}, \frac{5\pi}{4}\right)$



19. Rectangular coordinates: $(-6, 0)$

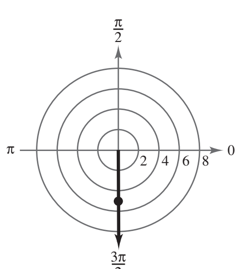
$r = \pm 6$, $\tan\theta = 0$, $\theta = 0$ or π

Polar coordinates: $(6, \pi), (-6, 0)$

20. Rectangular coordinates: $(0, -5)$

$r = 5$, $\tan\theta$ undefined, $\theta = \frac{\pi}{2}$

Polar coordinates: $\left(5, \frac{3\pi}{2}\right)$



21. Rectangular coordinates: $(-3, 4)$

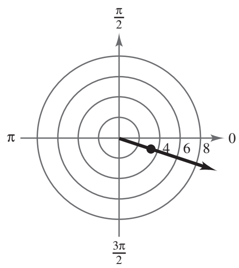
$r = \pm\sqrt{9 + 16} = \pm 5$, $\tan\theta = -\frac{4}{3}$, $\theta \approx 2.2143, 5.3559$

Polar coordinates: $(5, 2.2143), (-5, 5.3559)$

22. Rectangular coordinates: $(3, -1)$

$r = \sqrt{9+1} = \sqrt{10}$, $\tan\theta = -\frac{1}{3}$, $\theta \approx -0.322$ or 5.9614

Polar coordinates: $\left(\sqrt{10}, 5.961\right)$



23. Rectangular coordinates: $\left(-\sqrt{3}, -\sqrt{3}\right)$

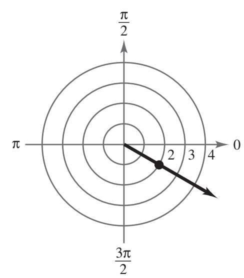
$r = \pm\sqrt{3+3} = \pm\sqrt{6}$, $\tan\theta = 1$, $\theta = \dfrac{\pi}{4}$ or $\dfrac{5\pi}{4}$

Polar coordinates: $\left(\sqrt{6}, \dfrac{5\pi}{4}\right), \left(-\sqrt{6}, \dfrac{\pi}{4}\right)$

24. Rectangular coordinates: $\left(\sqrt{3}, -1\right)$

$r = \sqrt{3+1} = 2$, $\tan\theta = -\dfrac{1}{\sqrt{3}}$, $\theta = \dfrac{11\pi}{6}$

Polar coordinates: $\left(2, \dfrac{11\pi}{6}\right)$



25. Rectangular coordinates: $(6, 9)$

$r = \pm\sqrt{6^2+9^2} = \pm\sqrt{117} = \pm 3\sqrt{13}$

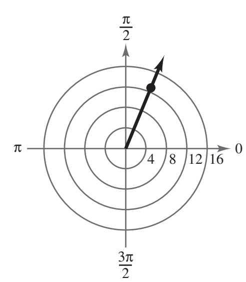
$\tan\theta = \frac{9}{6}$, $\theta \approx 0.9828, 4.1244$

Polar coordinates: $\left(3\sqrt{13}, 0.9828\right), \left(-3\sqrt{13}, 4.1244\right)$

26. Rectangular coordinates: $(5, 12)$

$r = \sqrt{25+144} = 13$, $\tan\theta = \frac{12}{5}$, $\theta \approx 1.176$

Polar coordinates: $(13, 1.176)$



27. Rectangular: $(3, -2)$

$(3, -2) \blacktriangleright$ Pol

$\approx (3.606, -0.5880)$

or $\left(\sqrt{13}, -0.5880\right)$

or $\left(\sqrt{13}, 5.6952\right)$

28. Rectangular coordinates: $(-5, 2)$

$R \blacktriangleright Pr(-5, 2) \approx 5.385$

$R \blacktriangleright P\theta(-5, 2) \approx 2.761$

$\approx (5.385, 2.761)$

29. Rectangular: $\left(\sqrt{3}, 2\right)$

$\left(\sqrt{3}, 2\right) \blacktriangleright$ Pol

$\approx (2.646, 0.8571)$

or $\left(\sqrt{7}, 0.8571\right)$

30. Rectangular coordinates: $\left(3\sqrt{2}, 3\sqrt{2}\right)$

$R \blacktriangleright Pr\left(3\sqrt{2}, 3\sqrt{2}\right) = 6$

$R \blacktriangleright P\theta\left(3\sqrt{2}, 3\sqrt{2}\right) \approx 0.785$

$= \left(6, \dfrac{\pi}{4}\right)$

31. Rectangular: $\left(\dfrac{5}{2}, \dfrac{4}{3}\right)$

$\left(\dfrac{5}{2}, \dfrac{4}{3}\right) \blacktriangleright$ Pol

$\approx (2.833, 0.4900)$

or $\left(\dfrac{17}{6}, 0.4900\right)$

32. Rectangular coordinates: $\left(\frac{7}{4}, \frac{3}{2}\right)$

$R \blacktriangleright Pr\left(\frac{7}{4}, \frac{3}{2}\right) \approx 2.305$

$R \blacktriangleright P\theta\left(\frac{7}{4}, \frac{3}{2}\right) \approx 0.709$

$\approx (2.305, 0.709)$

33. $x^2 + y^2 = 9$

$r = 3$

34. $x^2 + y^2 = 16$

$r = 4$

35.
$$\begin{aligned} y &= 4 \\ r \sin \theta &= 4 \\ r &= 4 \csc \theta \end{aligned}$$

36.
$$\begin{aligned} y &= x \\ r \cos \theta &= r \sin \theta \\ 1 &= \tan \theta \\ \theta &= \frac{\pi}{4} \end{aligned}$$

37.
$$\begin{aligned} x &= 10 \\ r \cos \theta &= 10 \\ r &= 10 \sec \theta \end{aligned}$$

38.
$$\begin{aligned} x &= 4a \\ r \cos \theta &= 4a \\ r &= 4a \sec \theta \end{aligned}$$

39.
$$\begin{aligned} 3x - y + 2 &= 0 \\ 3r \cos \theta - r \sin \theta + 2 &= 0 \\ r(3 \cos \theta - \sin \theta) &= -2 \\ r &= \frac{-2}{3 \cos \theta - \sin \theta} \end{aligned}$$

40.
$$\begin{aligned} 3x + 5y - 2 &= 0 \\ 3r \cos \theta + 5r \sin \theta - 2 &= 0 \\ r(3 \cos \theta + 5 \sin \theta) &= 2 \\ r &= \frac{2}{3 \cos \theta + 5 \sin \theta} \end{aligned}$$

41.
$$\begin{aligned} xy &= 16 \\ (r \cos \theta)(r \sin \theta) &= 16 \\ r^2 &= 16 \sec \theta \csc \theta = 32 \csc 2\theta \end{aligned}$$

42.
$$\begin{aligned} 2xy &= 1 \\ 2(r \cos \theta)(r \sin \theta) &= 1 \\ 2r^2 \cos \theta \sin \theta &= 1 \\ r^2 &= \frac{1}{2 \cos \theta \sin \theta} \\ r^2 &= \frac{1}{2} \sec \theta \csc \theta \\ r^2 &= \csc 2\theta \end{aligned}$$

43. $y^2 - 8x - 16 = 0$

$r^2 \sin^2 \theta - 8r \cos \theta - 16 = 0$

By the Quadratic Formula, we have:

$$\begin{aligned} r &= \frac{-(-8 \cos \theta) \pm \sqrt{(-8 \cos \theta)^2 - 4(\sin^2 \theta)(-16)}}{2 \sin^2 \theta} \\ &= \frac{8 \cos \theta \pm \sqrt{64 \cos^2 \theta + 64 \sin^2 \theta}}{2 \sin^2 \theta} \\ &= \frac{8 \cos \theta \pm \sqrt{64 (\cos^2 \theta + \sin^2 \theta)}}{2 \sin^2 \theta} \\ &= \frac{8 \cos \theta \pm 8}{2 \sin^2 \theta} \\ &= \frac{4(\cos \theta \pm 1)}{1 - \cos^2 \theta} \end{aligned}$$

$$r = \frac{4(\cos \theta + 1)}{(1 + \cos \theta)(1 - \cos \theta)} = \frac{4}{1 - \cos \theta}$$

or

$$r = \frac{4(\cos \theta - 1)}{(1 + \cos \theta)(1 - \cos \theta)} = \frac{-4}{1 + \cos \theta}$$

44.
$$\begin{aligned} (x^2 + y^2)^2 &= 9(x^2 - y^2) \\ (r^2)^2 &= 9(r^2 \cos^2 \theta - r^2 \sin^2 \theta) \\ &= 9r^2(\cos^2 \theta - \sin^2 \theta) \\ r^2 &= 9 \cos 2\theta \end{aligned}$$

45. $x^2 + y^2 = a^2$

$r^2 = a^2$

$r = a$

46. $x^2 + y^2 = 9a^2$

$r = 3a$

47. $x^2 + y^2 - 2ax = 0$

$r^2 - 2a\,r\cos\theta = 0$

$r(r - 2a\cos\theta) = 0$

$r - 2a\cos\theta = 0$

$r = 2a\cos\theta$

48. $x^2 + y^2 - 2ay = 0$

$r^2 - 2ar\sin\theta = 0$

$r = 2a\sin\theta$

49. $r = 4\sin\theta$

$r^2 = 4r\sin\theta$

$x^2 + y^2 = 4y$

$x^2 + y^2 - 4y = 0$

50. Because $x = r\cos\theta$ and r is given as $2\cos\theta$:

$x = 2\cos\theta\cos\theta = 2\cos^2\theta$

$r = 2\cos\theta$

$r^2 = 4\cos^2\theta$

$x^2 + y^2 = 2\,(2\cos^2\theta)$

$x^2 + y^2 = 2x$

$x^2 + y^2 - 2x = 0$

51. $\theta = \dfrac{2\pi}{3}$

$\tan\theta = \tan\dfrac{2\pi}{3}$

$\dfrac{y}{x} = -\sqrt{3}$

$y = -\sqrt{3}x$

$\sqrt{3}x + y = 0$

52. $\theta = \dfrac{5\pi}{3}$

$\tan\theta = -\sqrt{3}$

$\dfrac{y}{x} = -\sqrt{3}$

$y = -\sqrt{3}\,x$

$\sqrt{3}\,x + y = 0$

53. $r = 4$

$r^2 = 16$

$x^2 + y^2 = 16$

54. $r = 10$

$r^2 = 100$

$x^2 + y^2 = 100$

55. $r = 4\csc\theta$

$r\sin\theta = 4$

$y = 4$

56. $r = -3\sec\theta$

$\dfrac{r}{\sec\theta} = -3$

$r\cos\theta = -3$

$x = -3$

57. $r^2 = \cos\theta$

$r^3 = r\cos\theta$

$\left(\pm\sqrt{x^2 + y^2}\right)^3 = x$

$\pm\left(x^2 + y^2\right)^{3/2} = x$

$(x^2 + y^2)^3 = x^2$

$x^2 + y^2 = x^{2/3}$

$x^2 + y^2 - x^{2/3} = 0$

58. $r^2 = \sin 2\theta = 2\sin\theta\cos\theta$

$r^2 = 2\left(\dfrac{y}{r}\right)\left(\dfrac{x}{r}\right) = \dfrac{2xy}{r^2}$

$r^4 = 2xy$

$(x^2 + y^2)^2 = 2xy$

59. $r = 2\sin 3\theta$

$r = 2\sin(\theta + 2\theta)$

$r = 2[\sin\theta\cos 2\theta + \cos\theta\sin 2\theta]$

$r = 2[\sin\theta(1 - 2\sin^2\theta) + \cos\theta(2\sin\theta\cos\theta)]$

$r = 2[\sin\theta - 2\sin^3\theta + 2\sin\theta\cos^2\theta]$

$r = 2[\sin\theta - 2\sin^3\theta + 2\sin\theta(1 - \sin^2\theta)]$

$r = 2(3\sin\theta - 4\sin^3\theta)$

$r^4 = 6r^3\sin\theta - 8r^3\sin^3\theta$

$\left(x^2 + y^2\right)^2 = 6\left(x^2 + y^2\right)y - 8y^3$

$\left(x^2 + y^2\right)^2 = 6x^2y - 2y^3$

60.
$$\begin{aligned}
r &= 3\cos 2\theta\\
r &= 3(2\cos^2\theta - 1)\\
r &= 3\left(\frac{2x^2}{r^2} - 1\right)\\
r &= 3\left(\frac{2x^2 - r^2}{r^2}\right)\\
r &= 3\left(\frac{2x^2 - x^2 - y^2}{x^2 + y^2}\right)\\
r &= 3\left(\frac{x^2 - y^2}{x^2 + y^2}\right)\\
r^2 &= 9\left(\frac{x^2 - y^2}{x^2 + y^2}\right)^2\\
x^2 + y^2 &= 9\frac{(x^2 - y^2)^2}{(x^2 + y^2)^2}\\
(x^2 + y^2)^3 &= 9(x^2 - y^2)^2
\end{aligned}$$

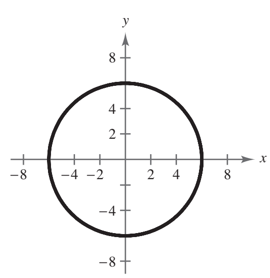
61.
$$\begin{aligned}
r &= \frac{2}{1 + \sin\theta}\\
r(1 + \sin\theta) &= 2\\
r + r\sin\theta &= 2\\
r &= 2 - r\sin\theta\\
\pm\sqrt{x^2 + y^2} &= 2 - y\\
x^2 + y^2 &= (2 - y)^2\\
x^2 + y^2 &= 4 - 4y + y^2\\
x^2 + 4y - 4 &= 0
\end{aligned}$$

62.
$$\begin{aligned}
r &= \frac{1}{1 - \cos\theta}\\
r - r\cos\theta &= 1\\
\sqrt{x^2 + y^2} - x &= 1\\
x^2 + y^2 &= 1 + 2x + x^2\\
y^2 &= 2x + 1
\end{aligned}$$

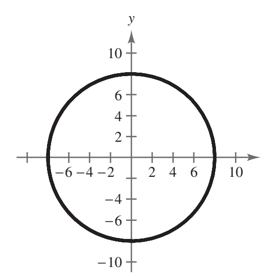
63.
$$\begin{aligned}
r &= \frac{6}{2 - 3\sin\theta}\\
r(2 - 3\sin\theta) &= 6\\
2r &= 6 + 3r\sin\theta\\
2\left(\pm\sqrt{x^2 + y^2}\right) &= 6 + 3y\\
4\left(x^2 + y^2\right) &= (6 + 3y)^2\\
4x^2 + 4y^2 &= 36 + 36y + 9y^2\\
4x^2 - 5y^2 - 36y - 36 &= 0
\end{aligned}$$

64.
$$\begin{aligned}
r &= \frac{6}{2\cos\theta - 3\sin\theta}\\
r &= \frac{6}{2(x/r) - 3(y/r)}\\
r &= \frac{6r}{2x - 3y}\\
1 &= \frac{6}{2x - 3y}\\
2x - 3y &= 6
\end{aligned}$$

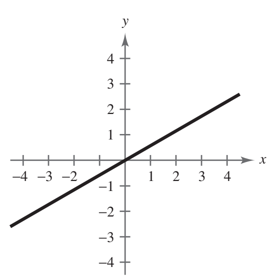
65. The graph of the polar equation consists of all points that are six units from the pole.
$$\begin{aligned}
r &= 6\\
r^2 &= 36\\
x^2 + y^2 &= 36
\end{aligned}$$



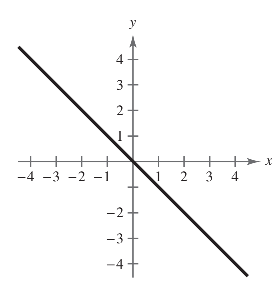
66. The graph of the polar equation consists of all points that are eight units from the pole.
$$\begin{aligned}
r &= 8\\
r^2 &= 64\\
x^2 + y^2 &= 64
\end{aligned}$$



67. The graph of the polar equation consists of all points that make an angle of $\pi/6$ with the polar axis.
$$\begin{aligned}
\theta &= \frac{\pi}{6}\\
\tan\theta &= \tan\frac{\pi}{6}\\
\frac{y}{x} &= \frac{\sqrt{3}}{3}\\
y &= \frac{\sqrt{3}}{3}x\\
3y &= \sqrt{3}x\\
-\sqrt{3}x + 3y &= 0
\end{aligned}$$

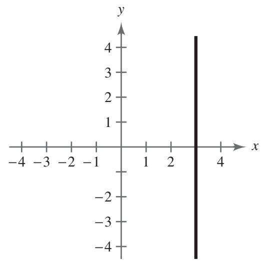


68. The graph of the polar equation consists of all points that make an angle of $3\pi/4$ with the polar axis.
$$\begin{aligned}
\theta &= \frac{3\pi}{4}\\
\tan\theta &= \tan\frac{3\pi}{4}\\
\frac{y}{x} &= -1\\
y &= -x\\
x + y &= 0
\end{aligned}$$



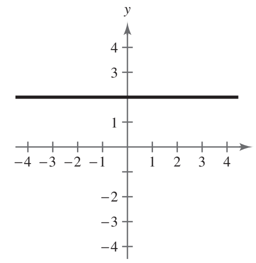
69. The graph of the polar equation is not evident by simple inspection. Convert to rectangular form first.

$$r = 3 \sec \theta$$
$$r \cos\theta = 3$$
$$x = 3$$
$$x - 3 = 0$$



70. The graph of the polar equation is not evident by simple inspection. Convert to rectangular form first.

$$r = 2 \csc \theta$$
$$r \sin \theta = 2$$
$$y = 2$$
$$y - 2 = 0$$



71. True. Because r is a directed distance, then the point (r, θ) can be represented as $(r, \theta \pm 2n\pi)$.

72. False. (r_1, θ) and (r_2, θ) represent the same point only if $r_1 = r_2$.

73.

$$\begin{aligned} r &= 2(h \cos \theta + k \sin \theta) \\ r &= 2\left(h\left(\frac{x}{r}\right) + k\left(\frac{y}{r}\right)\right) \\ r &= \frac{2hx + 2ky}{r} \\ r^2 &= 2hx + 2ky \\ x^2 + y^2 &= 2hx + 2ky \\ x^2 - 2hx + y^2 - 2ky &= 0 \\ \left(x^2 - 2hx + h^2\right) + \left(y^2 - 2ky + k^2\right) &= h^2 + k^2 \\ (x - h)^2 + (y - k)^2 &= h^2 + k^2 \end{aligned}$$

Center: (h, k)

Radius: $\sqrt{h^2 + k^2}$

74.

$$\begin{aligned} r &= \cos \theta + 3 \sin \theta \\ r &= \frac{x}{r} + \frac{3y}{r} \\ r^2 &= x + 3y \\ x^2 + y^2 &= x + 3y \\ x^2 - x + y^2 - 3y &= 0 \\ \left(x - \frac{1}{2}\right)^2 + \left(y - \frac{3}{2}\right)^2 &= \frac{5}{2} \end{aligned}$$

The graph is a circle.

75. (a) $(r_1, \theta_1) = (x_1, y_1)$ where $x_1 = r_1 \cos \theta_1$ and $y_1 = r_1 \sin \theta_1$.

$(r_2, \theta_2) = (x_2, y_2)$ where $x_2 = r_2 \cos \theta_2$ and $y_2 = r_2 \sin \theta_2$.

$$\begin{aligned} d &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= \sqrt{x_1^2 - 2x_1x_2 + x_2^2 + y_1^2 - 2y_1y_2 + y_2^2} \\ &= \sqrt{(x_1^2 + y_1^2) + (x_2^2 + y_2^2) - 2(x_1x_2 + y_1y_2)} \\ &= \sqrt{r_1^2 + r_2^2 - 2\left(r_1r_2 \cos \theta_1 \cos \theta_2 + r_1r_2 \sin \theta_1 \sin \theta_2\right)} \\ &= \sqrt{r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_1 - \theta_2)} \end{aligned}$$

(b) If $\theta_1 = \theta_2$, then

$$\begin{aligned} d &= \sqrt{r_1^2 + r_2^2 - 2r_1r_2} \\ &= \sqrt{(r_1 - r_2)^2} \\ &= |r_1 - r_2|. \end{aligned}$$

This represents the distance between two points on the line $\theta = \theta_1 = \theta_2$.

(c) If $\theta_1 - \theta_2 = 90°$, then

$d = \sqrt{r_1^2 + r_2^2}$.

This is the result of the Pythagorean Theorem.

(d) The results should be the same. For example, use the points

$$\left(3, \frac{\pi}{6}\right) \text{ and } \left(4, \frac{\pi}{3}\right).$$

The distance is $d \approx 2.053$.
Now use the representations

$$\left(-3, \frac{7\pi}{6}\right) \text{ and } \left(-4, \frac{4\pi}{3}\right).$$

The distance is still $d \approx 2.053$.

76. (a) For horizontal moves, just the x-coordinate changes. For vertical moves, just the y-coordinate changes.

(b) For horizontal moves, both r and θ change. For vertical moves, both r and θ change.

(c) Unlike r and θ, x and y measure horizontal and vertical changes, respectively.

77. $\log_6 \dfrac{x^2z}{3y} = \log_6 x^2z - \log_6 3y$

$= \log_6 x^2 + \log_6 z - (\log_6 3 + \log_6 y)$

$= 2\log_6 x + \log_6 z - \log_6 3 - \log_6 y$

78. $\log_4 \dfrac{\sqrt{2x}}{y} = \log_4 \sqrt{2x} - \log_4 y$

$= \dfrac{1}{2}\log_4 2x - \log_4 y$

$= \dfrac{1}{2}\log_4 2 + \dfrac{1}{2}\log_4 x - \log_4 y$

$= \dfrac{1}{4} + \dfrac{1}{2}\log_4 x - \log_4 y$

79. $\ln x(x+4)^2 = \ln x + \ln(x+4)^2$

$= \ln x + 2\ln(x+4)$

80. $\ln 5x^2(x^2+1) = \ln 5 + \ln x^2 + \ln(x^2+1)$

$= \ln 5 + 2\ln x + \ln(x^2+1)$

81. $\log_7 x - \log_7 3y = \log_7 \dfrac{x}{3y}$

82. $\log_5 a + 8\log_5(x+1) = \log_5 a + \log_5(x+1)^8$

$= \log_5 a(x+1)^8$

83. $\frac{1}{2}\ln x + \ln(x-2) = \ln\sqrt{x} + \ln(x-2)$

$= \ln\sqrt{x}(x-2)$

84. $\ln 6 + \ln y - \ln(x-3) = \ln 6y - \ln(x-3)$

$= \ln\dfrac{6y}{x-3}$

85. $\begin{cases} 5x - 7y = -11 \\ -3x + y = -3 \end{cases}$

By Cramer's Rule we have:

$$x = \frac{\begin{vmatrix} -11 & -7 \\ -3 & 1 \end{vmatrix}}{\begin{vmatrix} 5 & -7 \\ -3 & 1 \end{vmatrix}} = \frac{-32}{-16} = 2$$

$$y = \frac{\begin{vmatrix} 5 & -11 \\ -3 & -3 \end{vmatrix}}{\begin{vmatrix} 5 & -7 \\ -3 & 1 \end{vmatrix}} = \frac{-48}{-16} = 3$$

Solution: $(2, 3)$

86. $\begin{cases} 3x - 5y = 10 \\ 4x - 2y = -5 \end{cases}$

$$x = \frac{\begin{vmatrix} 10 & -5 \\ -5 & -2 \end{vmatrix}}{\begin{vmatrix} 3 & -5 \\ 4 & -2 \end{vmatrix}} = -\frac{45}{14}$$

$$y = \frac{\begin{vmatrix} 3 & 10 \\ 4 & -5 \end{vmatrix}}{\begin{vmatrix} 3 & -5 \\ 4 & -2 \end{vmatrix}} = -\frac{55}{14}$$

Solution: $\left(-\dfrac{45}{14}, -\dfrac{55}{14}\right)$

87. $\begin{cases} 3a - 2b + c = 0 \\ 2a + b - 3c = 0 \\ a - 3b + 9c = 8 \end{cases}$

$$\begin{vmatrix} 3 & -2 & 1 \\ 2 & 1 & -3 \\ 1 & -3 & 9 \end{vmatrix} = 35$$

By Cramer's Rule we have:

$$a = \frac{\begin{vmatrix} 0 & -2 & 1 \\ 0 & 1 & -3 \\ 8 & -3 & 9 \end{vmatrix}}{35} = \frac{40}{35} = \frac{8}{7}$$

$$b = \frac{\begin{vmatrix} 3 & 0 & 1 \\ 2 & 0 & -3 \\ 1 & 8 & 9 \end{vmatrix}}{35} = \frac{88}{35}$$

$$c = \frac{\begin{vmatrix} 3 & -2 & 0 \\ 2 & 1 & 0 \\ 1 & -3 & 8 \end{vmatrix}}{35} = \frac{56}{35} = \frac{8}{5}$$

Solution: $\left(\frac{8}{7}, \frac{88}{35}, \frac{8}{5}\right)$

88. $\begin{cases} 5u + 7v + 9w = 15 \\ u - 2v - 3w = 7 \\ 8u - 2v + w = 0 \end{cases} \Rightarrow \begin{vmatrix} 5 & 7 & 9 \\ 1 & -2 & -3 \\ 8 & -2 & 1 \end{vmatrix} = -89$

$$u = \frac{\begin{vmatrix} 15 & 7 & 9 \\ 7 & -2 & -3 \\ 0 & -2 & 1 \end{vmatrix}}{-89} = \frac{-295}{-89} = \frac{295}{89},$$

$$v = \frac{\begin{vmatrix} 5 & 15 & 9 \\ 1 & 7 & -3 \\ 8 & 0 & 1 \end{vmatrix}}{-89} = \frac{-844}{-89} = \frac{844}{89},$$

$$w = \frac{\begin{vmatrix} 5 & 7 & 15 \\ 1 & -2 & 7 \\ 8 & -2 & 0 \end{vmatrix}}{-89} = \frac{672}{-89} = -\frac{672}{89}$$

Solution: $\left(\frac{295}{89}, \frac{844}{89}, -\frac{672}{89}\right)$

89. $\begin{cases} -x + y + 2z = 1 \\ 2x + 3y + z = -2 \\ 5x + 4y + 2z = 4 \end{cases}$

$$\begin{vmatrix} -1 & 1 & 2 \\ 2 & 3 & 1 \\ 5 & 4 & 2 \end{vmatrix} = -15$$

By Cramer's Rule we have:

$$x = \frac{\begin{vmatrix} 1 & 1 & 2 \\ -2 & 3 & 1 \\ 4 & 4 & 2 \end{vmatrix}}{-15} = \frac{-30}{-15} = 2$$

$$y = \frac{\begin{vmatrix} -1 & 1 & 2 \\ 2 & -2 & 1 \\ 5 & 4 & 2 \end{vmatrix}}{-15} = \frac{45}{-15} = -3$$

$$z = \frac{\begin{vmatrix} -1 & 1 & 1 \\ 2 & 3 & -2 \\ 5 & 4 & 4 \end{vmatrix}}{-15} = \frac{-45}{-15} = 3$$

Solution: $(2, -3, 3)$

90. $\begin{cases} 2x_1 + x_2 + 2x_3 = 4 \\ 2x_1 + 2x_2 = 5 \\ 2x_1 - x_2 + 6x_3 = 2 \end{cases}$

$$D = \begin{vmatrix} 2 & 1 & 2 \\ 2 & 2 & 0 \\ 2 & -1 & 6 \end{vmatrix} = 0$$

Cramer's Rule does not apply.

91. Points: $(4, -3), (6, -7), (-2, -1)$

$$\begin{vmatrix} 4 & -3 & 1 \\ 6 & -7 & 1 \\ -2 & -1 & 1 \end{vmatrix} = -20 \neq 0$$

The points are not collinear.

92. Points: $(-2, 4), (0, 1), (4, -5)$

$$\begin{vmatrix} -2 & 4 & 1 \\ 0 & 1 & 1 \\ 4 & -5 & 1 \end{vmatrix} = 0 \Rightarrow \text{collinear}$$

93. Points: $(-6, -4), (-1, -3), (1.5, -2.5)$

$$\begin{vmatrix} -6 & -4 & 1 \\ -1 & -3 & 1 \\ 1.5 & -2.5 & 1 \end{vmatrix} = 0$$

The points are collinear.

94. Points: $(-2.3, 5), (-0.5, 0), (1.5, -3)$

$$\begin{vmatrix} -2.3 & 5 & 1 \\ -0.5 & 0 & 1 \\ 1.5 & -3 & 1 \end{vmatrix} = 4.6 \implies \text{not collinear}$$

Section 10.8 Graphs of Polar Equations

- When graphing polar equations:
  1. Test for symmetry.
     (a) $\theta = \pi/2$: Replace (r, θ) by $(r, \pi - \theta)$ or $(-r, -\theta)$.
     (b) Polar axis: Replace (r, θ) by $(r, -\theta)$ or $(-r, \pi - \theta)$.
     (c) Pole: Replace (r, θ) by $(r, \pi + \theta)$ or $(-r, \theta)$.
     (d) $r = f(\sin\theta)$ is symmetric with respect to the line $\theta = \pi/2$.
     (e) $r = f(\cos\theta)$ is symmetric with respect to the polar axis.
  2. Find the θ values for which $|r|$ is maximum.
  3. Find the θ values for which $r = 0$.
  4. Know the different types of polar graphs.
     (a) Limaçons $(0 < a, 0 < b)$: $r = a \pm b\cos\theta$, $r = a \pm b\sin\theta$
     (b) Rose curves, $n \geq 2$: $r = a\cos n\theta$, $r = a\sin n\theta$
     (c) Circles: $r = a\cos\theta$, $r = a\sin\theta$, $r = a$
     (d) Lemniscates: $r^2 = a^2\cos 2\theta$, $r^2 = a^2\sin 2\theta$
  5. Plot additional points.

Vocabulary Check

1. $\theta = \dfrac{\pi}{2}$ **2.** polar axis **3.** convex limaçon

4. circle **5.** lemniscate **6.** cardioid

1. $r = 3\cos 2\theta$
Rose curve with 4 petals

2. $r = 5 - 5\sin\theta$
Cardioid

3. $r = 3(1 - 2\cos\theta)$
Limaçon with inner loop

4. $r^2 = 16\cos 2\theta$
Lemniscate

5. $r = 6\sin 2\theta$
Rose curve with 4 petals

6. $r = 3\cos\theta$
Circle

7. $r = 5 + 4\cos\theta$

$\theta = \dfrac{\pi}{2}$: $-r = 5 + 4\cos(-\theta)$
$-r = 5 + 4\cos\theta$
Not an equivalent equation

Polar axis: $r = 5 + 4\cos(-\theta)$
$r = 5 + 4\cos\theta$
Equivalent equation

Pole: $-r = 5 + 4\cos\theta$
Not an equivalent equation

Answer: Symmetric with respect to polar axis

8. $r = 16\cos 3\theta$

$\theta = \dfrac{\pi}{2}$: $-r = 16\cos(3(-\theta))$
$-r = 16\cos(-3\theta)$
$-r = 16\cos 3\theta$
Not an equivalent equation

Polar axis: $r = 16\cos(3(-\theta))$
$r = 16\cos(-3\theta)$
$r = 16\cos 3\theta$
Equivalent equation

Pole: $-r = 16\cos 3\theta$
Not an equivalent equation

Answer: Symmetric with respect to polar axis

9. $r = \dfrac{2}{1 + \sin\theta}$

$\theta = \dfrac{\pi}{2}$: $r = \dfrac{2}{1 + \sin(\pi - \theta)}$

$r = \dfrac{2}{1 + \sin\pi\cos\theta - \cos\pi\sin\theta}$

$r = \dfrac{2}{1 + \sin\theta}$

Equivalent equation

Polar axis: $r = \dfrac{2}{1 + \sin(-\theta)}$

$r = \dfrac{2}{1 - \sin\theta}$

Not an equivalent equation

Pole: $-r = \dfrac{2}{1 + \sin\theta}$

Answer: Symmetric with respect to $\theta = \pi/2$

10. $r = \dfrac{3}{2 + \cos\theta}$

$\theta = \dfrac{\pi}{2}$: $-r = \dfrac{3}{2 + \cos(-\theta)}$

Not an equivalent equation

Polar axis: $r = \dfrac{3}{2 + \cos(-\theta)}$

Equivalent equation

Pole: $-r = \dfrac{3}{2 + \cos\theta}$

Not an equivalent equation

Answer: Symmetric with respect to polar axis

11. $r^2 = 16\cos 2\theta$

$\theta = \dfrac{\pi}{2}$: $(-r)^2 = 16\cos 2(-\theta)$

$r^2 = 16\cos 2\theta$

Equivalent equation

Polar axis: $r^2 = 16\cos 2(-\theta)$

$r^2 = 16\cos 2\theta$

Equivalent equation

Pole: $(-r)^2 = 16\cos 2\theta$

$r^2 = 16\cos 2\theta$

Equivalent equation

Answer: Symmetric with respect to $\theta = \dfrac{\pi}{2}$, the polar axis, and the pole

12. $r^2 = 36\sin 2\theta$

$\theta = \dfrac{\pi}{2}$: $(-r)^2 = 36\sin(-2\theta)$

Not an equivalent equation

Polar axis: $r^2 = 36\sin(-2\theta)$

Not an equivalent equation

Pole: $(-r)^2 = 36\sin 2\theta$

Equivalent equation

Answer: Symmetric with respect to pole

13. $|r| = |10(1 - \sin\theta)| = 10|1 - \sin\theta| \le 10(2) = 20$

$|1 - \sin\theta| = 2$

$1 - \sin\theta = 2$ or $1 - \sin\theta = -2$

$\sin\theta = -1$ $\qquad$ $\sin\theta = 3$

$\theta = \dfrac{3\pi}{2}$ $\qquad$ Not possible

Maximum: $|r| = 20$ when $\theta = \dfrac{3\pi}{2}$

$0 = 10(1 - \sin\theta)$

$\sin\theta = 1$

$\theta = \dfrac{\pi}{2}$

Zero: $r = 0$ when $\theta = \dfrac{\pi}{2}$

14. $|r| = |6 + 12\cos\theta| \le |6| + |12\cos\theta|$

$= 6 + 12|\cos\theta| \le 18$

$\cos\theta = 1$

$\theta = 0$

Maximum: $|r| = 18$ when $\theta = 0$

$0 = 6 + 12\cos\theta$

$\cos\theta = -\dfrac{1}{2}$

$\theta = \dfrac{2\pi}{3}, \dfrac{4\pi}{3}$

Zero: $r = 0$ when $\theta = \dfrac{2\pi}{3}, \dfrac{4\pi}{3}$

15. $|r| = |4\cos 3\theta| = 4|\cos 3\theta| \le 4$

$|\cos 3\theta| = 1$

$\cos 3\theta = \pm 1$

$\theta = 0, \dfrac{\pi}{3}, \dfrac{2\pi}{3}$

Maximum: $|r| = 4$ when $\theta = 0, \dfrac{\pi}{3}, \dfrac{2\pi}{3}$

$0 = 4\cos 3\theta$

$\cos 3\theta = 0$

$\theta = \dfrac{\pi}{6}, \dfrac{\pi}{2}, \dfrac{5\pi}{6}$

Zero: $r = 0$ when $\theta = \dfrac{\pi}{6}, \dfrac{\pi}{2}, \dfrac{5\pi}{6}$

16. $|r| = |3\sin 2\theta| = 3|\sin 2\theta| \le 3$

$|\sin 2\theta| = 1$

$\sin 2\theta = \pm 1$

$\theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$

Maximum: $|r| = 3$ when $\theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$

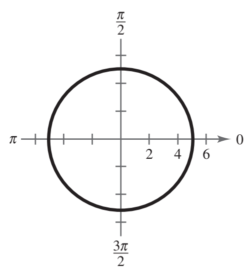
$0 = 3\sin 2\theta$

$\sin 2\theta = 0$

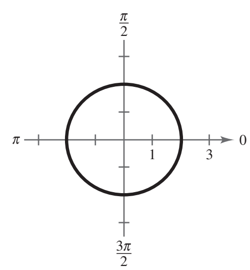
$\theta = 0, \dfrac{\pi}{2}, \pi, \dfrac{3\pi}{2}$

Zero: $r = 0$ when $\theta = 0, \dfrac{\pi}{2}, \pi, \dfrac{3\pi}{2}$

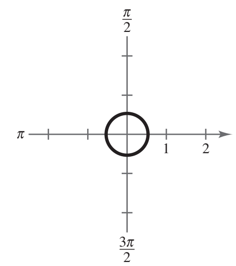
17. Circle: $r = 5$



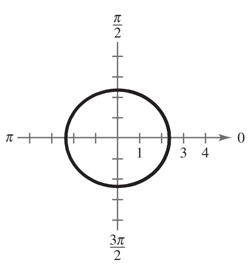
18. Circle: $r = 2$



19. Circle: $r = \dfrac{\pi}{6}$



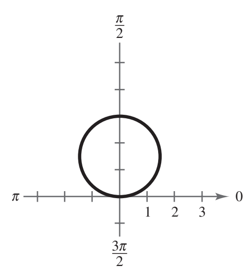
20. $r = -\dfrac{3\pi}{4}$



21. $r = 3\sin\theta$

Symmetric with respect to $\theta = \pi/2$

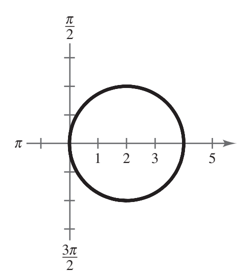
Circle with a radius of $3/2$



22. $r = 4\cos\theta$

Symmetric with respect to polar axis

Circle with radius 2



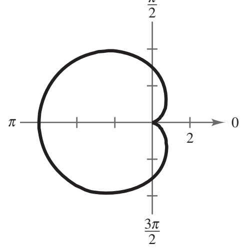
23. $r = 3(1 - \cos\theta)$

Symmetric with respect to the polar axis

$\dfrac{a}{b} = \dfrac{3}{3} = 1 \implies$ Cardioid

$|r| = 6$ when $\theta = \pi$

$r = 0$ when $\theta = 0$



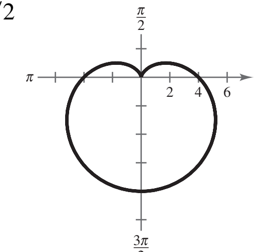
24. $r = 4(1 - \sin\theta)$

Symmetric with respect to $\pi/2$

$\dfrac{a}{b} = \dfrac{4}{4} = 1 \implies$ Cardioid

$|r| = 8$ when $\theta = \dfrac{3\pi}{2}$

$r = 0$ when $\theta = \dfrac{\pi}{2}$



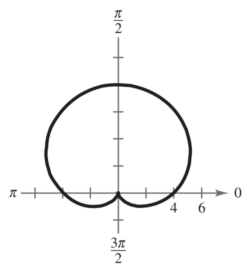
25. $r = 4(1 + \sin\theta)$

Symmetric with respect to $\theta = \dfrac{\pi}{2}$

$\dfrac{a}{b} = \dfrac{4}{4} = 1 \implies$ Cardioid

$|r| = 8$ when $\theta = \dfrac{\pi}{2}$

$r = 0$ when $\theta = \dfrac{3\pi}{2}$



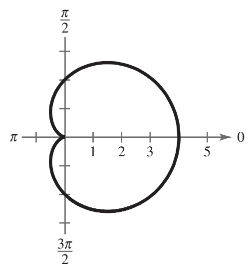
26. $r = 2(1 + \cos\theta)$

Symmetric with respect to polar axis

$\dfrac{a}{b} = \dfrac{2}{2} = 1 \implies$ Cardioid

$|r| = 4$ when $\theta = 0$

$r = 0$ when $\theta = \pi$



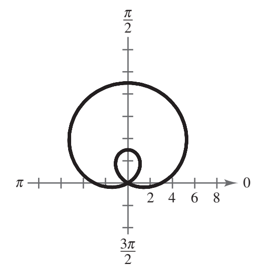
27. $r = 3 + 6\sin\theta$

Symmetric with respect to $\theta = \dfrac{\pi}{2}$

$\dfrac{a}{b} = \dfrac{3}{6} < 1 \implies$ Limaçon with inner loop

$|r| = 9$ when $\theta = \dfrac{\pi}{2}$

$r = 0$ when $\theta = \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$



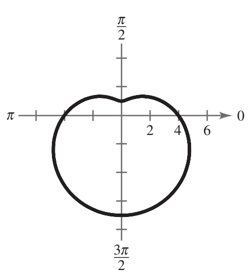
28. $r = 4 - 3\sin\theta$

Symmetric with respect to $\pi/2$

$a = 4, b = 3$

$\dfrac{a}{b} = \dfrac{4}{3} \implies$ Dimpled limaçon

$|r| = 7$ when $\theta = \dfrac{3\pi}{2}$



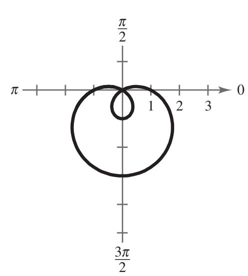
29. $r = 1 - 2\sin\theta$

Symmetric with respect to $\theta = \dfrac{\pi}{2}$

$\dfrac{a}{b} = \dfrac{1}{2} < 1 \implies$ Limaçon with inner loop

$|r| = 3$ when $\theta = \dfrac{3\pi}{2}$

$r = 0$ when $\theta = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$



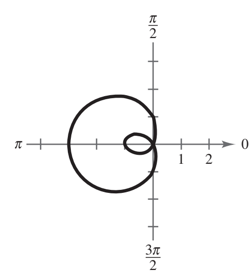
30. $r = 1 - 2\cos\theta$

Symmetric with respect to the polar axis

$\dfrac{a}{b} = \dfrac{1}{2} \implies$ Limaçon with inner loop

$|r| = 3$ when $\theta = \pi$

$r = 0$ when $\theta = \dfrac{\pi}{3}, \dfrac{5\pi}{3}$



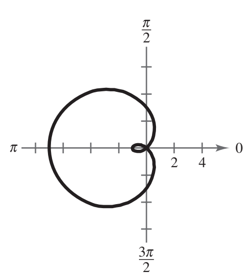
31. $r = 3 - 4\cos\theta$

Symmetric with respect to the polar axis

$\dfrac{a}{b} = \dfrac{3}{4} < 1 \implies$ Limaçon with inner loop

$|r| = 7$ when $\theta = \pi$

$r = 0$ when $\cos\theta = \dfrac{3}{4}$ or $\theta \approx 0.723, 5.560$

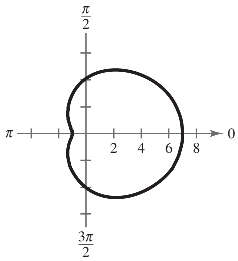


32. $r = 4 + 3\cos\theta$

Symmetric with respect to the polar axis

$\dfrac{a}{b} = \dfrac{4}{3} > 1 \implies$ Dimpled limaçon

$|r| = 7$ when $\theta = 0$



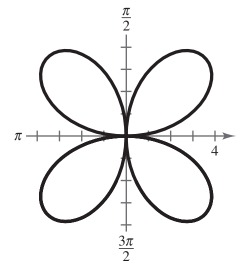
33. $r = 5\sin 2\theta$

Symmetric with respect to $\theta = \pi/2$, the polar axis, and the pole

Rose curve $(n = 2)$ with 4 petals

$|r| = 5$ when $\theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$

$r = 0$ when $\theta = 0, \dfrac{\pi}{2}, \pi$



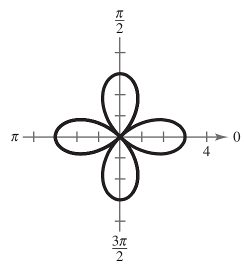
34. $r = 3\cos 2\theta$

Symmetric with respect to the polar axis

Rose curve $(n = 2)$ with four petals

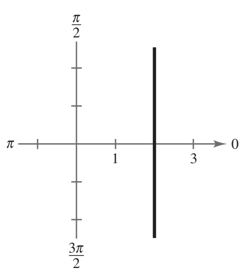
$|r| = 3$ when $\theta = 0, \dfrac{\pi}{2}, \pi, \dfrac{3\pi}{2}$

$r = 0$ when $\theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$



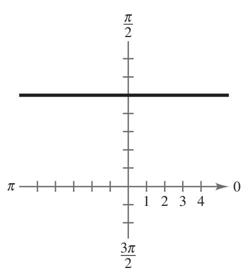
35.

$$r = 2\sec\theta$$
$$r = \frac{2}{\cos\theta}$$
$$r\cos\theta = 2$$
$$x = 2 \implies \text{Line}$$



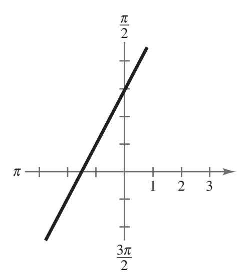
36.

$$r = 5\csc\theta$$
$$r\sin\theta = 5$$
$$y = 5 \implies \text{Line}$$



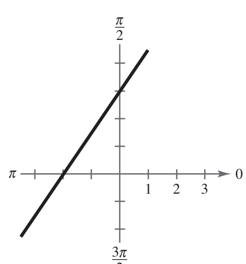
37.

$$r = \frac{3}{\sin\theta - 2\cos\theta}$$
$$r(\sin\theta - 2\cos\theta) = 3$$
$$y - 2x = 3$$
$$y = 2x + 3 \implies \text{Line}$$



38.

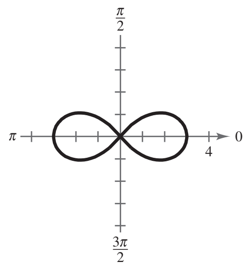
$$r = \frac{6}{2\sin\theta - 3\cos\theta}$$
$$r(2\sin\theta - 3\cos\theta) = 6$$
$$2y - 3x = 6$$
$$y = \frac{3}{2}x + 3 \implies \text{Line}$$



39. $r^2 = 9\cos 2\theta$

Symmetric with respect to the polar axis, $\theta = \pi/2$, and the pole

Lemniscate

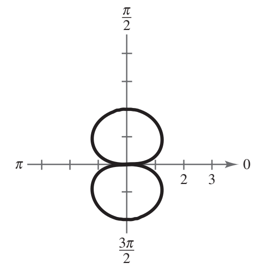


40. $r^2 = 4\sin\theta$

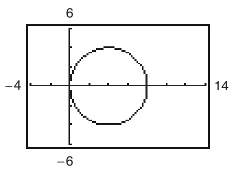
$r = 2\sqrt{\sin\theta}$

$r = -2\sqrt{\sin\theta}$

$0 \le \theta \le \pi$



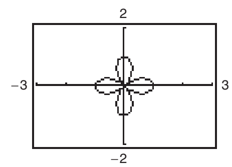
41. $r = 8\cos\theta$



$0 \le \theta \le 2\pi$

θmin = 0
θmax = 2π
θstep = $\pi/24$
Xmin = -4
Xmax = 14
Xscl = 2
Ymin = -6
Ymax = 6
Yscl = 2

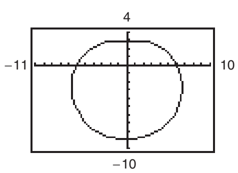
42. $r = \cos 2\theta$



$0 \le \theta \le 2\pi$

θmin = 0
θmax = 2π
θstep = $\pi/24$
Xmin = -3
Xmax = 3
Xscl = 1
Ymin = -2
Ymax = 2
Yscl = 1

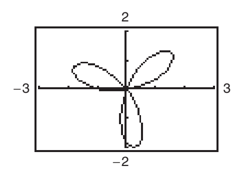
43. $r = 3(2 - \sin\theta)$



$0 \le \theta \le 2\pi$

θmin = 0
θmax = 2π
θstep = $\pi/24$
Xmin = -10
Xmax = 10
Xscl = 1
Ymin = -10
Ymax = 4
Yscl = 1

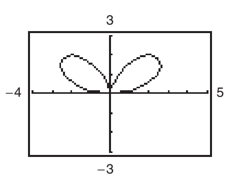
44. $r = 2\cos(3\theta - 2)$



$0 \le \theta \le \pi$

θmin = 0
θmax = π
θstep = $\pi/24$
Xmin = -2
Xmax = 2
Xscl = 1
Ymin = -2
Ymax = 2
Yscl = 1

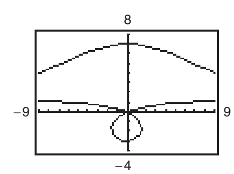
45. $r = 8\sin\theta\cos^2\theta$



$0 \le \theta \le 2\pi$

θmin = 0
θmax = 2π
θstep = $\pi/24$
Xmin = -4
Xmax = 4
Xscl = 1
Ymin = -3
Ymax = 3
Yscl = 1

46. $r = 2\csc\theta + 5 = \dfrac{2}{\sin\theta} + 5$

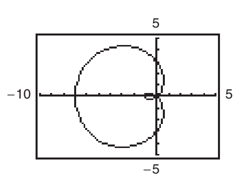


$0 \le \theta \le 2\pi$

θmin = 0
θmax = 2π
θstep = $\pi/24$
Xmin = -9
Xmax = 9
Xscl = 1
Ymin = -4
Ymax = 8
Yscl = 1

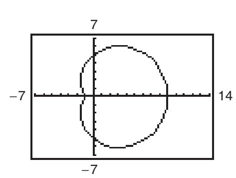
47. $r = 3 - 4\cos\theta$

$0 \le \theta < 2\pi$



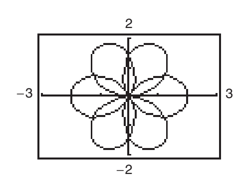
48. $r = 5 + 4\cos\theta$

$0 \le \theta < 2\pi$



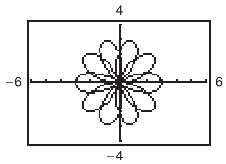
49. $r = 2\cos\left(\dfrac{3\theta}{2}\right)$

$0 \le \theta < 4\pi$



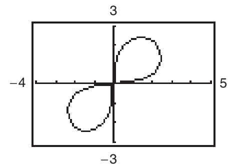
50. $r = 3\sin\left(\dfrac{5\theta}{2}\right)$

$0 \le \theta < 4\pi$



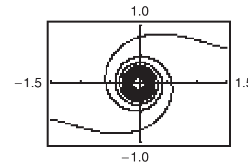
51. $r^2 = 9\sin 2\theta$

$0 \le \theta < \pi$



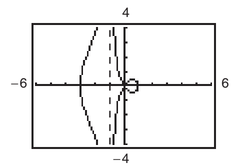
52. $r^2 = \dfrac{1}{\theta}$

$0 < \theta < \infty$



53.

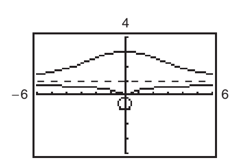
$$\begin{aligned}
r &= 2 - \sec\theta = 2 - \frac{1}{\cos\theta}\\
r\cos\theta &= 2\cos\theta - 1\\
r(r\cos\theta) &= 2r\cos\theta - r\\
\left(\pm\sqrt{x^2+y^2}\right)x &= 2x - \left(\pm\sqrt{x^2+y^2}\right)\\
\left(\pm\sqrt{x^2+y^2}\right)(x+1) &= 2x\\
\left(\pm\sqrt{x^2+y^2}\right) &= \frac{2x}{x+1}\\
x^2+y^2 &= \frac{4x^2}{(x+1)^2}\\
y^2 &= \frac{4x^2}{(x+1)^2} - x^2\\
&= \frac{4x^2 - x^2(x+1)^2}{(x+1)^2} = \frac{4x^2 - x^2\left(x^2+2x+1\right)}{(x+1)^2}\\
&= \frac{-x^4 - 2x^3 + 3x^2}{(x+1)^2} = \frac{-x^2\left(x^2+2x-3\right)}{(x+1)^2}\\
y &= \pm\sqrt{\frac{x^2\left(3-2x-x^2\right)}{(x+1)^2}} = \pm\left|\frac{x}{x+1}\right|\sqrt{3-2x-x^2}
\end{aligned}$$



The graph has an asymptote at $x = -1$.

54.

$$\begin{aligned}
r &= 2 + \csc\theta = 2 + \frac{1}{\sin\theta}\\
r\sin\theta &= 2\sin\theta + 1\\
r(r\sin\theta) &= 2r\sin\theta + r\\
\left(\pm\sqrt{x^2+y^2}\right)(y) &= 2y + \left(\pm\sqrt{x^2+y^2}\right)\\
\left(\pm\sqrt{x^2+y^2}\right)(y-1) &= 2y\\
\left(\pm\sqrt{x^2+y^2}\right) &= \frac{2y}{y-1}\\
x^2+y^2 &= \frac{4y^2}{(y-1)^2}\\
x^2 &= \frac{y^2(3+2y-y^2)}{(y-1)^2}\\
x &= \pm\sqrt{\frac{y^2(3+2y-y^2)}{(y-1)^2}} = \pm\left|\frac{y}{y-1}\right|\sqrt{3+2y-y^2}
\end{aligned}$$



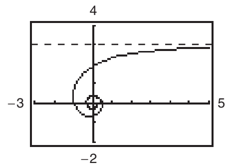
The graph has an asymptote at $y = 1$.

55. $r = \dfrac{3}{\theta}$

$\theta = \dfrac{3}{r} = \dfrac{3 \sin \theta}{r \sin \theta} = \dfrac{3 \sin \theta}{y}$

$y = \dfrac{3 \sin \theta}{\theta}$

As $\theta \to 0, y \to 3$



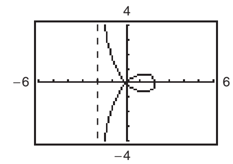
56. $r = 2 \cos 2\theta \sec \theta = \dfrac{2 \cos 2\theta}{\cos \theta}$

$r = \dfrac{2(\cos^2 \theta - \sin^2 \theta)}{\cos \theta}$

$r \cos \theta = 2(\cos^2 \theta - \sin^2 \theta)$

$x = 2(\cos^2 \theta - \sin^2 \theta)$

As $\theta \to \dfrac{\pi}{2}, x \to -2.$



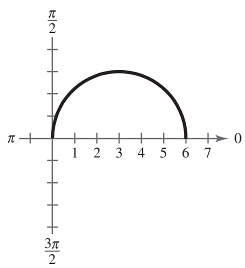
57. True. For a graph to have polar axis symmetry, replace (r, θ) by $(r, -\theta)$ or $(-r, \pi - \theta)$.

58. False. For a graph to be symmetric about the pole, one portion of the graph coincides with the other portion when rotated π radians about the pole.

59. $r = 6 \cos \theta$

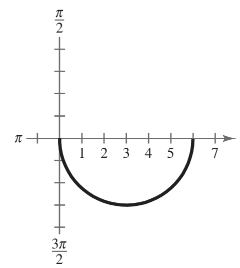
(a) $0 \le \theta \le \dfrac{\pi}{2}$

Upper half of circle



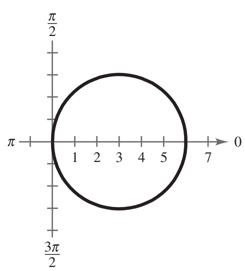
(b) $\dfrac{\pi}{2} \le \theta \le \pi$

Lower half of circle



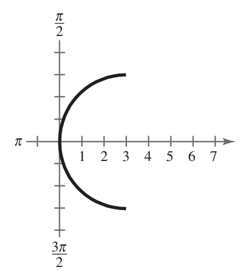
(c) $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$

Entire circle



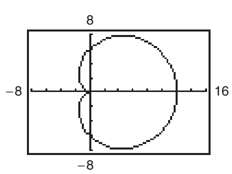
(d) $\dfrac{\pi}{4} \le \theta \le \dfrac{3\pi}{4}$

Left half of circle

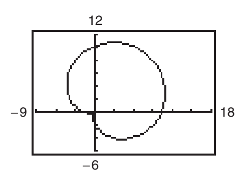


60. $r = 6[1 + \cos(\theta - \phi)]$

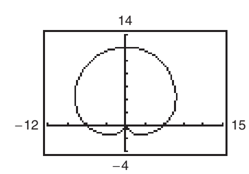
(a)



(b)



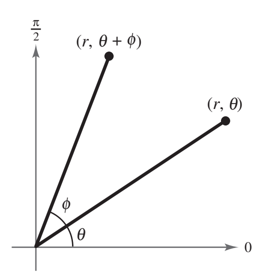
(c)



The angle ϕ has the effect of rotating the graph by the angle ϕ. For part (c),

$$r = 6\left[1 + \cos\left(\theta - \frac{\pi}{2}\right)\right] = 6(1 + \sin \theta).$$

61. Let the curve $r = f(\theta)$ be rotated by ϕ to form the curve $r = g(\theta)$. If (r_1, θ_1) is a point on $r = f(\theta)$, then $(r_1, \theta_1 + \phi)$ is on $r = g(\theta)$. That is, $g(\theta_1 + \phi) = r_1 = f(\theta_1)$. Letting $\theta = \theta_1 + \phi$, or $\theta_1 = \theta - \phi$, we see that $g(\theta) = g(\theta_1 + \phi) = f(\theta_1) = f(\theta - \phi)$.



62. Use the result of Exercise 61.

(a) Rotation: $\phi = \dfrac{\pi}{2}$

Original graph: $r = f(\sin\theta)$

Rotated graph: $r = f\left(\sin\left(\theta - \dfrac{\pi}{2}\right)\right) = f(-\cos\theta)$

(b) Rotation: $\phi = \pi$

Original graph: $r = f(\sin\theta)$

Rotated graph: $r = f(\sin(\theta - \pi)) = f(-\sin\theta)$

(c) Rotation: $\phi = \dfrac{3\pi}{2}$

Original graph: $r = f(\sin\theta)$

Rotated graph: $r = f\left(\sin\left(\theta - \dfrac{3\pi}{2}\right)\right) = f(\cos\theta)$

63. (a) $$\begin{aligned} r &= 2 - \sin\left(\theta - \frac{\pi}{4}\right) \\ &= 2 - \left[\sin\theta\cos\frac{\pi}{4} - \cos\theta\sin\frac{\pi}{4}\right] \\ &= 2 - \frac{\sqrt{2}}{2}(\sin\theta - \cos\theta) \end{aligned}$$

(c) $$\begin{aligned} r &= 2 - \sin(\theta - \pi) \\ &= 2 - [\sin\theta\cos\pi - \cos\theta\sin\pi] \\ &= 2 + \sin\theta \end{aligned}$$

(b) $$\begin{aligned} r &= 2 - \sin\left(\theta - \frac{\pi}{2}\right) \\ &= 2 - \left[\sin\theta\cos\frac{\pi}{2} - \cos\theta\sin\frac{\pi}{2}\right] \\ &= 2 + \cos\theta \end{aligned}$$

(d) $$\begin{aligned} r &= 2 - \sin\left(\theta - \frac{3\pi}{2}\right) \\ &= 2 - \left[\sin\theta\cos\frac{3\pi}{2} - \cos\theta\sin\frac{3\pi}{2}\right] \\ &= 2 - \cos\theta \end{aligned}$$

64. $r = 2\sin 2\theta$

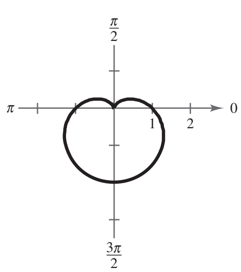
(a) $$\begin{aligned} r &= 2\sin\left[2\left(\theta - \frac{\pi}{6}\right)\right] \\ &= 2\left[2\sin\left(\theta - \frac{\pi}{6}\right)\cos\left(\theta - \frac{\pi}{6}\right)\right] \\ &= 4\sin\left(\theta - \frac{\pi}{6}\right)\cos\left(\theta - \frac{\pi}{6}\right) \end{aligned}$$

(c) $$\begin{aligned} r &= 2\sin\left[2\left(\theta - \frac{2\pi}{3}\right)\right] \\ &= 2\left[2\sin\left(\theta - \frac{2\pi}{3}\right)\cos\left(\theta - \frac{2\pi}{3}\right)\right] \\ &= 4\sin\left(\theta - \frac{2\pi}{3}\right)\cos\left(\theta - \frac{2\pi}{3}\right) \end{aligned}$$

(b) $$\begin{aligned} r &= 2\sin\left[2\left(\theta - \frac{\pi}{2}\right)\right] \\ &= 2\sin(2\theta - \pi) \\ &= -2\sin 2\theta \\ &= -2(2\sin\theta\cos\theta) \\ &= -4\sin\theta\cos\theta \end{aligned}$$

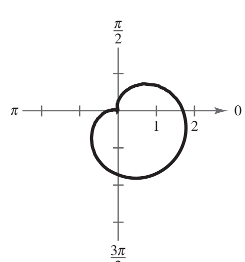
(d) $$\begin{aligned} r &= 2\sin[2(\theta - \pi)] \\ &= 2\sin(2\theta - 2\pi) \\ &= 2\sin 2\theta \\ &= 2[2\sin\theta\cos\theta] \\ &= 4\sin\theta\cos\theta \end{aligned}$$

65. (a) $r = 1 - \sin\theta$

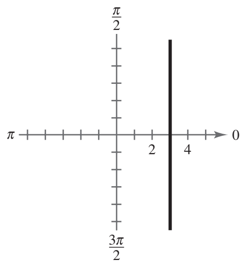


(b) $r = 1 - \sin\left(\theta - \dfrac{\pi}{4}\right)$

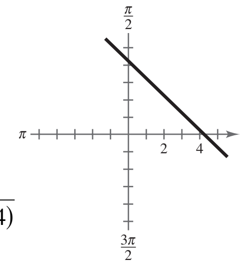
Rotate the graph in part (a) through the angle $\dfrac{\pi}{4}$.



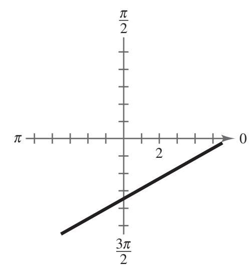
66. (a)
$$r = 3 \sec \theta$$
$$r = \frac{3}{\cos \theta}$$
$$r \cos \theta = 3 \implies x = 3$$



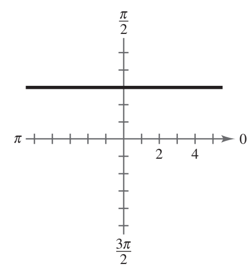
(b)
$$r = 3 \sec\left(\theta - \frac{\pi}{4}\right)$$
$$r = \frac{3}{\cos(\theta - (\pi/4))}$$
$$r = \frac{3}{\cos \theta \cos(\pi/4) + \sin \theta \sin(\pi/4)}$$
$$\frac{\sqrt{2}}{2} r \cos \theta + \frac{\sqrt{2}}{2} r \sin \theta = 3$$
$$\frac{\sqrt{2}}{2} x + \frac{\sqrt{2}}{2} y = 3$$



(c)
$$r = 3 \sec\left(\theta + \frac{\pi}{3}\right)$$
$$r = \frac{3}{\cos(\theta + (\pi/3))}$$
$$r = \frac{3}{\cos \theta \cos(\pi/3) - \sin \theta \sin(\pi/3)}$$
$$\frac{1}{2} r \cos \theta - \frac{\sqrt{3}}{2} r \sin \theta = 3$$
$$\frac{1}{2} x - \frac{\sqrt{3}}{2} y = 3$$



(d)
$$r = 3 \sec\left(\theta - \frac{\pi}{2}\right)$$
$$r = \frac{3}{\cos(\theta - (\pi/2))}$$
$$r = \frac{3}{\cos \theta \cos(\pi/2) + \sin \theta \sin(\pi/2)}$$
$$r \sin \theta = 3 \implies y = 3$$



67. $r = 2 + k \sin \theta$

$k = 0$: $r = 2$

Circle

$k = 1$: $r = 2 + \sin \theta$

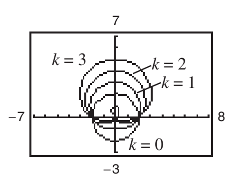
Convex limaçon

$k = 2$: $r = 2 + 2 \sin \theta$

Cardioid

$k = 3$: $r = 2 + 3 \sin \theta$

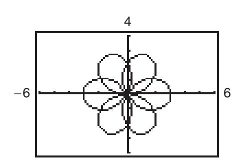
Limaçon with inner loop



68. $r = 3 \sin k\theta$

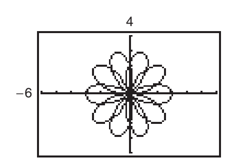
(a) $r = 3 \sin 1.5\theta$

$0 \le \theta < 4\pi$



(b) $r = 3 \sin 2.5\theta$

$0 \le \theta < 4\pi$



(c) Yes. $r = 3 \sin(k\theta)$.

Find the minimum value of θ, $(\theta > 0)$, that is a multiple of 2π that makes $k\theta$ a multiple of 2π.

69. $y = \dfrac{x^2 - 9}{x + 1}$

$$\frac{x^2 - 9}{x + 1} = 0$$
$$x^2 - 9 = 0$$
$$x^2 = 9$$
$$x = \pm 3$$

70. $y = 6 + \dfrac{4}{x^2 + 4}$

No zeros

71. $y = 5 - \dfrac{3}{x - 2}$

$$5 - \frac{3}{x - 2} = 0$$
$$5 = \frac{3}{x - 2}$$
$$5(x - 2) = 3$$
$$5x - 10 = 3$$
$$5x = 13$$
$$x = \frac{13}{5}$$

72. $y = \dfrac{x^3 - 27}{x^2 + 4}$

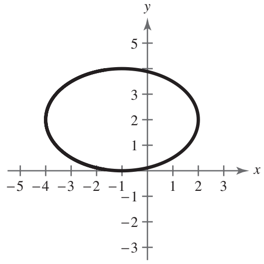
Zero: $x = 3$

73. Vertices: $(-4, 2), (2, 2) \implies$ Center at $(-1, 2)$ and $a = 3$

Minor axis of length 4: $2b = 4 \implies b = 2$

Horizontal major axis

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$$
$$\frac{(x + 1)^2}{9} + \frac{(y - 2)^2}{4} = 1$$



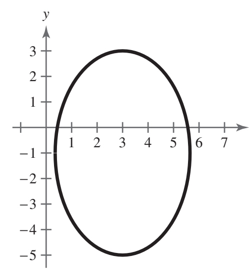
74. Foci: $(3, 2), (3, -4)$; Major axis of length 8

Center: $(h, k) = (3, -1)$

Vertical major axis

$a = 4, c = 3, b^2 = a^2 - c^2 = 16 - 9 \implies b = \sqrt{7}$

$$\frac{(x - h)^2}{b^2} + \frac{(y - k)^2}{a^2} = 1$$
$$\frac{(x - 3)^2}{7} + \frac{(y + 1)^2}{16} = 1$$



Section 10.9 Polar Equations of Conics

- The graph of a polar equation of the form

  $$r = \frac{ep}{1 \pm e\cos\theta} \quad \text{or} \quad r = \frac{ep}{1 \pm e\sin\theta}$$

  is a conic, where $e > 0$ is the eccentricity and $|p|$ is the distance between the focus (pole) and the directrix.

  (a) If $e < 1$, the graph is an ellipse.
  (b) If $e = 1$, the graph is a parabola.
  (c) If $e > 1$, the graph is a hyperbola.

- Guidelines for finding polar equations of conics:

  (a) Horizontal directrix above the pole: $r = \dfrac{ep}{1 + e\sin\theta}$

  (b) Horizontal directrix below the pole: $r = \dfrac{ep}{1 - e\sin\theta}$

  (c) Vertical directrix to the right of the pole: $r = \dfrac{ep}{1 + e\cos\theta}$

  (d) Vertical directrix to the left of the pole: $r = \dfrac{ep}{1 - e\cos\theta}$

Vocabulary Check

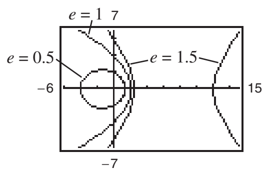
1. conic
2. eccentricity; e
3. vertical; right
4. (a) iii (b) i (c) ii

1. $r = \dfrac{4e}{1 + e\cos\theta}$

(a) $e = 1$, $r = \dfrac{4}{1 + \cos\theta}$, parabola

(b) $e = 0.5$, $r = \dfrac{2}{1 + 0.5\cos\theta} = \dfrac{4}{2 + \cos\theta}$, ellipse

(c) $e = 1.5$, $r = \dfrac{6}{1 + 1.5\cos\theta} = \dfrac{12}{2 + 3\cos\theta}$, hyperbola

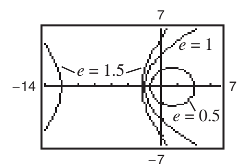


2. $r = \dfrac{4e}{1 - e\cos\theta}$

$e = 1$, $r = \dfrac{4}{1 - \cos\theta}$, parabola

$e = 0.5$, $r = \dfrac{2}{1 - 0.5\cos\theta}$, ellipse

$e = 1.5$, $r = \dfrac{6}{1 - 1.5\cos\theta}$, hyperbola

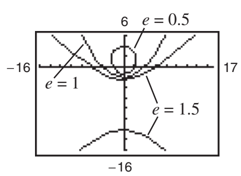


3. $r = \dfrac{4e}{1 - e\sin\theta}$

(a) $e = 1$, $r = \dfrac{4}{1 - \sin\theta}$, parabola

(b) $e = 0.5$, $r = \dfrac{2}{1 - 0.5\sin\theta} = \dfrac{4}{2 - \sin\theta}$, ellipse

(c) $e = 1.5$, $r = \dfrac{6}{1 - 1.5\sin\theta} = \dfrac{12}{2 - 3\sin\theta}$, hyperbola

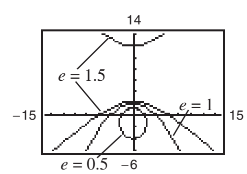


4. $r = \dfrac{4e}{1 + e\sin\theta}$

$e = 1$, $r = \dfrac{4}{1 + \sin\theta}$, parabola

$e = 0.5$, $r = \dfrac{2}{1 + 0.5\sin\theta}$, ellipse

$e = 1.5$, $r = \dfrac{6}{1 + 1.5\sin\theta}$, hyperbola



5. $r = \dfrac{2}{1 + \cos\theta}$

$e = 1 \implies$ Parabola

Vertical directrix to the right of the pole

Matches graph (f).

6. $r = \dfrac{3}{2 - \cos\theta}$

$e = \dfrac{1}{2} \implies$ Ellipse

Vertical directrix to the left of the pole

Matches graph (c).

7. $r = \dfrac{3}{1 + 2\sin\theta}$

$e = 2 \implies$ Hyperbola

Matches graph (d).

8. $r = \dfrac{2}{1 - \sin\theta}$

$e = 1 \implies$ Parabola

Horizontal directrix below pole

Matches graph (e).

9. $r = \dfrac{4}{2 + \cos\theta}$

$= \dfrac{2}{1 + 0.5\cos\theta}$

$e = 0.5 \implies$ Ellipse

Matches graph (a).

10. $r = \dfrac{4}{1 - 3\sin\theta}$

$e = 3 \implies$ Hyperbola

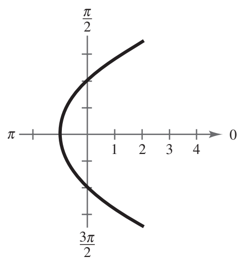
Horizontal directrix below pole

Matches graph (b).

11. $r = \dfrac{2}{1 - \cos\theta}$

$e = 1$, the graph is a parabola.

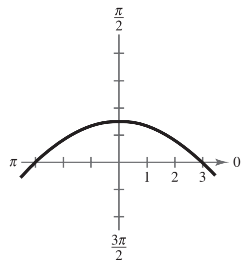
Vertex: $(1, \pi)$



12. $r = \dfrac{3}{1 + \sin\theta}$

$e = 1 \implies$ Parabola

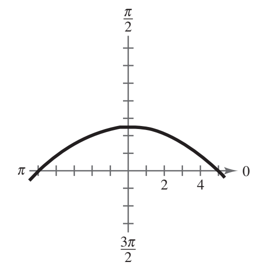
Vertex: $\left(\dfrac{3}{2}, \dfrac{\pi}{2}\right)$



13. $r = \dfrac{5}{1 + \sin\theta}$

$e = 1$, the graph is a parabola.

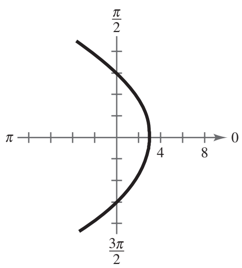
Vertex: $\left(\dfrac{5}{2}, \dfrac{\pi}{2}\right)$



14. $r = \dfrac{6}{1 + \cos\theta}$

$e = 1 \implies$ Parabola

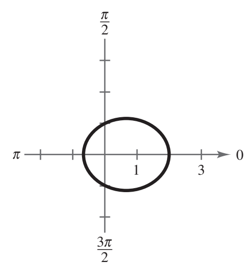
Vertex: $(3, 0)$



15. $r = \dfrac{2}{2 - \cos\theta} = \dfrac{1}{1 - (1/2)\cos\theta}$

$e = \dfrac{1}{2} < 1$, the graph is an ellipse.

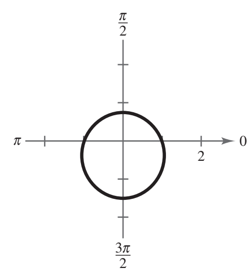
Vertices: $(2, 0), \left(\dfrac{2}{3}, \pi\right)$



16. $r = \dfrac{3}{3 + \sin\theta} = \dfrac{1}{1 + (1/3)\sin\theta}$

$e = \dfrac{1}{3} < 1 \implies$ Ellipse

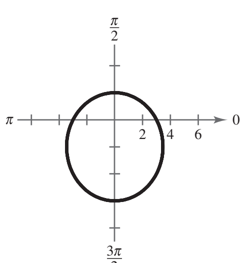
Vertices: $\left(\dfrac{3}{4}, \dfrac{\pi}{2}\right), \left(\dfrac{3}{2}, \dfrac{3\pi}{2}\right)$



17. $r = \dfrac{6}{2 + \sin\theta} = \dfrac{3}{1 + (1/2)\sin\theta}$

$e = \dfrac{1}{2} < 1$, the graph is an ellipse.

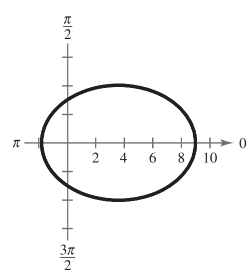
Vertices: $\left(2, \dfrac{\pi}{2}\right), \left(6, \dfrac{3\pi}{2}\right)$



18. $r = \dfrac{9}{3 - 2\cos\theta} = \dfrac{3}{1 - (2/3)\cos\theta}$

$e = \dfrac{2}{3} < 1 \implies$ Ellipse

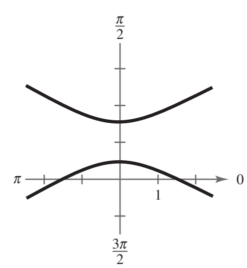
Vertices: $(9, 0), \left(\dfrac{9}{5}, \pi\right)$



19. $r = \dfrac{3}{2 + 4\sin\theta} = \dfrac{3/2}{1 + 2\sin\theta}$

$e = 2 > 1$, the graph is a hyperbola.

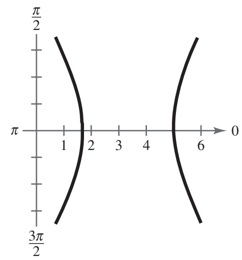
Vertices: $\left(\dfrac{1}{2}, \dfrac{\pi}{2}\right), \left(-\dfrac{3}{2}, \dfrac{3\pi}{2}\right)$



20. $r = \dfrac{5}{-1 + 2\cos\theta} = \dfrac{-5}{1 - 2\cos\theta}$

$e = 2 > 1 \implies$ Hyperbola

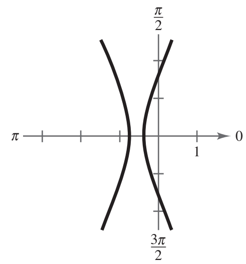
Vertices: $(5, 0), \left(-\dfrac{5}{3}, \pi\right)$



21. $r = \dfrac{3}{2 - 6\cos\theta} = \dfrac{3/2}{1 - 3\cos\theta}$

$e = 3 > 1$, the graph is a hyperbola.

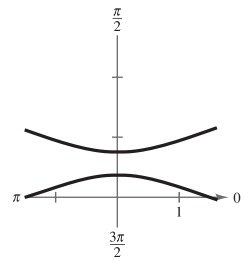
Vertices: $\left(-\dfrac{3}{4}, 0\right), \left(\dfrac{3}{8}, \pi\right)$



22. $r = \dfrac{3}{2 + 6\sin\theta} = \dfrac{3/2}{1 + 3\sin\theta}$

$e = 3 > 1 \implies$ Hyperbola

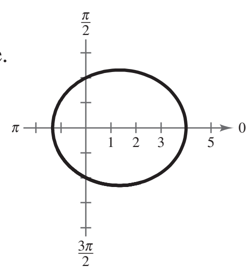
Vertices: $\left(\dfrac{3}{8}, \dfrac{\pi}{2}\right), \left(-\dfrac{3}{4}, \dfrac{3\pi}{2}\right)$



23. $r = \dfrac{4}{2 - \cos\theta} = \dfrac{2}{1 - (1/2)\cos\theta}$

$e = \dfrac{1}{2} < 1$, the graph is an ellipse.

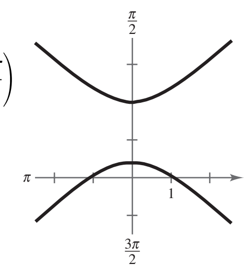
Vertices: $(4, 0), \left(\dfrac{4}{3}, \pi\right)$



24. $r = \dfrac{2}{2 + 3\sin\theta} = \dfrac{1}{1 + (3/2)\sin\theta}$

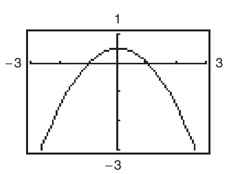
$e = \dfrac{3}{2} > 1 \implies$ Hyperbola

Vertices: $\left(\dfrac{2}{5}, \dfrac{\pi}{2}\right), \left(-2, \dfrac{3\pi}{2}\right)$



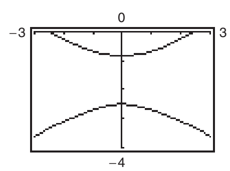
25. $r = \dfrac{-1}{1 - \sin\theta}$

$e = 1 \implies$ Parabola



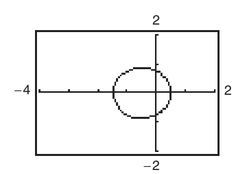
26. $r = \dfrac{-5}{2 + 4\sin\theta} = \dfrac{-(5/2)}{1 + 2\sin\theta}$

$e = 2 \implies$ Hyperbola



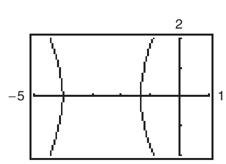
27. $r = \dfrac{3}{-4 + 2\cos\theta}$

$e = \dfrac{1}{2} \implies$ Ellipse



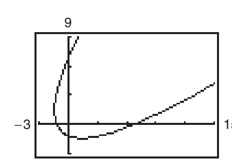
28. $r = \dfrac{4}{1 - 2\cos\theta}$

$e = 2 \implies$ Hyperbola



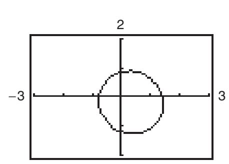
29. $r = \dfrac{2}{1 - \cos\left(\theta - \dfrac{\pi}{4}\right)}$

Rotate the graph in Exercise 11 through the angle $\pi/4$.



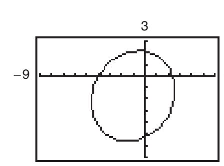
30. $r = \dfrac{3}{3 + \sin\left(\theta - \dfrac{\pi}{3}\right)}$

Rotate the graph in Exercise 16 through the angle $\pi/3$.

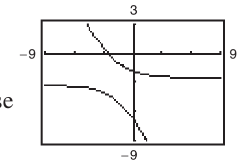


31. $r = \dfrac{6}{2 + \sin\left(\theta + \dfrac{\pi}{6}\right)}$

Rotate the graph in Exercise 17 through the angle $-\pi/6$.



32. $r = \dfrac{5}{-1 + 2\cos\left(\theta + \dfrac{2\pi}{3}\right)}$



Rotate the graph in Exercise 20 through the angle $-2\pi/3$.

33. Parabola: $e = 1$

Directrix: $x = -1$

Vertical directrix to the left of the pole

$$r = \frac{1(1)}{1 - 1\cos\theta} = \frac{1}{1 - \cos\theta}$$

34. Parabola: $e = 1$

Directrix: $y = -2$

$p = 2$

Horizontal directrix below the pole

$$r = \frac{1(2)}{1 - 1\sin\theta} = \frac{2}{1 - \sin\theta}$$

35. Ellipse: $e = \dfrac{1}{2}$

Directrix: $y = 1$

$p = 1$

Horizontal directrix above the pole

$$r = \frac{(1/2)(1)}{1 + (1/2)\sin\theta} = \frac{1}{2 + \sin\theta}$$

36. Ellipse: $e = \dfrac{3}{4}$

Directrix: $y = -3$

$p = 3$

Horizontal directrix below the pole

$$r = \frac{(3/4)(3)}{1 - (3/4)\sin\theta} = \frac{9}{4 - 3\sin\theta}$$

37. Hyperbola: $e = 2$

Directrix: $x = 1$

$p = 1$

Vertical directrix to the right of the pole

$$r = \frac{2(1)}{1 + 2\cos\theta} = \frac{2}{1 + 2\cos\theta}$$

38. Hyperbola: $e = \dfrac{3}{2}$

Directrix: $x = -1$

$p = 1$

Vertical directrix to the left of the pole

$$r = \frac{(3/2)(1)}{1 - (3/2)\cos\theta} = \frac{3}{2 - 3\cos\theta}$$

39. Parabola

Vertex: $\left(1, -\dfrac{\pi}{2}\right) \Longrightarrow e = 1, p = 2$

Horizontal directrix below the pole

$$r = \frac{1(2)}{1 - 1\sin\theta} = \frac{2}{1 - \sin\theta}$$

40. Parabola

Vertex: $(6, 0) \Longrightarrow e = 1, p = 12$

Vertical directrix to the right of the pole

$$r = \frac{1(12)}{1 + 1\cos\theta} = \frac{12}{1 + \cos\theta}$$

41. Parabola

Vertex: $(5, \pi) \Longrightarrow e = 1, p = 10$

Vertical directrix to the left of the pole

$$r = \frac{1(10)}{1 - 1\cos\theta} = \frac{10}{1 - \cos\theta}$$

42. Parabola

Vertex: $\left(10, \dfrac{\pi}{2}\right) \Longrightarrow e = 1, p = 20$

Horizontal directrix above the pole

$$r = \frac{1(20)}{1 + 1\sin\theta} = \frac{20}{1 + \sin\theta}$$

43. Ellipse: Vertices $(2, 0), (10, \pi)$

Center: $(4, \pi)$; $c = 4, a = 6, e = \dfrac{2}{3}$

Vertical directrix to the right of the pole

$$r = \frac{(2/3)p}{1 + (2/3)\cos\theta} = \frac{2p}{3 + 2\cos\theta}$$

$$2 = \frac{2p}{3 + 2\cos 0}$$

$p = 5$

$$r = \frac{2(5)}{3 + 2\cos\theta} = \frac{10}{3 + 2\cos\theta}$$

44. Ellipse

Vertices: $\left(2, \frac{\pi}{2}\right), \left(4, \frac{3\pi}{2}\right)$

Center: $\left(1, \frac{3\pi}{2}\right)$; $c = 1, a = 3, e = \frac{1}{3}$

Horizontal directrix above the axis

$$r = \frac{1/3p}{1 + (1/3)\sin\theta} = \frac{p}{3 + \sin\theta}$$

$$2 = \frac{p}{3 + \sin(\pi/2)}$$

$$p = 8$$

$$r = \frac{8}{3 + \sin\theta}$$

45. Ellipse: Vertices $(20, 0), (4, \pi)$

Center: $(8, 0)$; $c = 8, a = 12, e = \frac{2}{3}$

Vertical directrix to the left of the pole

$$r = \frac{(2/3)p}{1 - (2/3)\cos\theta} = \frac{2p}{3 - 2\cos\theta}$$

$$20 = \frac{2p}{3 - 2\cos 0}$$

$$p = 10$$

$$r = \frac{2(10)}{3 - 2\cos\theta} = \frac{20}{3 - 2\cos\theta}$$

46. Hyperbola

Vertices: $(2, 0), (8, 0)$

Center: $(5, 0)$; $c = 5, a = 3, e = \frac{5}{3}$

Vertical directrix to the right of the pole

$$r = \frac{(5/3)p}{1 + (5/3)\cos\theta} = \frac{5p}{3 + 5\cos\theta}$$

$$2 = \frac{5p}{3 + 5\cos 0}$$

$$p = \frac{16}{5}$$

$$r = \frac{5(16/5)}{3 + 5\cos\theta} = \frac{16}{3 + 5\cos\theta}$$

47. Hyperbola: Vertices $\left(1, \frac{3\pi}{2}\right), \left(9, \frac{3\pi}{2}\right)$

Center: $\left(5, \frac{3\pi}{2}\right)$; $c = 5, a = 4, e = \frac{5}{4}$

Horizontal directrix below the pole

$$r = \frac{(5/4)p}{1 - (5/4)\sin\theta} = \frac{5p}{4 - 5\sin\theta}$$

$$1 = \frac{5p}{4 - 5\sin(3\pi/2)}$$

$$p = \frac{9}{5}$$

$$r = \frac{5(9/5)}{4 - 5\sin\theta} = \frac{9}{4 - 5\sin\theta}$$

48. Hyperbola

Vertices: $\left(4, \frac{\pi}{2}\right), \left(1, \frac{\pi}{2}\right)$

Center: $\left(\frac{5}{2}, \frac{\pi}{2}\right)$; $c = \frac{5}{2}, a = \frac{3}{2}, e = \frac{5/2}{3/2} = \frac{5}{3}$

Horizontal directrix above the pole

$$r = \frac{(5/3)p}{1 + (5/3)\sin\theta} = \frac{5p}{3 + 5\sin\theta}$$

$$1 = \frac{5p}{3 + 5\sin(\pi/2)}$$

$$p = \frac{8}{5}$$

$$r = \frac{5(8/5)}{3 + 5\sin\theta} = \frac{8}{3 + 5\sin\theta}$$

49. When $\theta = 0$, $r = c + a = ea + a = a(1 + e)$.

Therefore,

$$a(1 + e) = \frac{ep}{1 - e\cos 0}$$

$$a(1 + e)(1 - e) = ep$$

$$a(1 - e^2) = ep.$$

Thus, $r = \dfrac{ep}{1 - e\cos\theta} = \dfrac{(1 - e^2)a}{1 - e\cos\theta}$.

50. Minimum distance occurs when $\theta = \pi$.

$$r = \frac{(1 - e^2)a}{1 - e\cos\pi} = \frac{(1 - e)(1 + e)a}{1 + e} = a(1 - e)$$

Maximum distance occurs when $\theta = 0$.

$$r = \frac{(1 - e^2)a}{1 - e\cos 0} = \frac{(1 - e)(1 + e)a}{1 - e} = a(1 + e)$$

51. $r = \dfrac{[1 - (0.0167)^2](95.956 \times 10^6)}{1 - 0.0167\cos\theta} \approx \dfrac{9.5929 \times 10^7}{1 - 0.0167\cos\theta}$

Perihelion distance:
$r = 95.956 \times 10^6(1 - 0.0167) \approx 9.4354 \times 10^7$ miles

Aphelion distance:
$r = 95.956 \times 10^6(1 + 0.0167) \approx 9.7558 \times 10^7$ miles

52. $r = \dfrac{[1 - (0.0542)^2](1.427 \times 10^9)}{1 - 0.0542\cos\theta} \approx \dfrac{1.4228 \times 10^9}{1 - 0.0542\cos\theta}$

Perihelion distance: $r = 1.427 \times 10^9(1 - 0.0542) \approx 1.3497 \times 10^9$ kilometers

Aphelion distance: $r = 1.427 \times 10^9(1 + 0.0542) \approx 1.5043 \times 10^9$ kilometers

53. $r = \dfrac{[1 - (0.0068)^2](108.209 \times 10^6)}{1 - 0.0068\cos\theta} \approx \dfrac{1.0820 \times 10^8}{1 - 0.0068\cos\theta}$

Perihelion distance: $r = 108.209 \times 10^6(1 - 0.0068) \approx 1.0747 \times 10^8$ kilometers

Aphelion distance: $r = 108.209 \times 10^6(1 + 0.0068) \approx 1.0894 \times 10^8$ kilometers

54. $r = \dfrac{[1 - (0.2056)^2](35.98 \times 10^6)}{1 - 0.2056\cos\theta} \approx \dfrac{3.4459 \times 10^7}{1 - 0.2056\cos\theta}$

Perihelion distance: $r = 35.98 \times 10^6(1 - 0.2056) \approx 2.8583 \times 10^7$ miles

Aphelion distance: $r = 35.98 \times 10^6(1 + 0.2056) \approx 4.3377 \times 10^7$ miles

55. $r = \dfrac{[1 - (0.0934)^2](141.63 \times 10^6)}{1 - 0.0934\cos\theta} \approx \dfrac{1.4039 \times 10^8}{1 - 0.0934\cos\theta}$

Perihelion distance: $r = 141.63 \times 10^6(1 - 0.0934) \approx 1.2840 \times 10^8$ miles

Aphelion distance: $r = 141.63 \times 10^6(1 + 0.0934) \approx 1.5486 \times 10^8$ miles

56. $r = \dfrac{[1 - (0.0484)^2](778.41 \times 10^6)}{1 - 0.0484\cos\theta} \approx \dfrac{7.7659 \times 10^8}{1 - 0.0484\cos\theta}$

Perihelion distance: $r = 778.41 \times 10^6(1 - 0.0484) \approx 7.4073 \times 10^8$ kilometers

Aphelion distance: $r = 778.41 \times 10^6(1 + 0.0484) \approx 8.1609 \times 10^8$ kilometers

57. $e \approx 0.847$, $a \approx \dfrac{4.42}{2} = 2.21$

$$2a = \frac{0.847p}{1 + 0.847} + \frac{0.847p}{1 - 0.847} \approx 5.9945p \approx 4.42$$

$p \approx 0.737$, $ep \approx 0.624$

$$r = \frac{0.624}{1 + 0.847\sin\theta}$$

To find the closest point to the sun, let $\theta = \dfrac{\pi}{2}$.

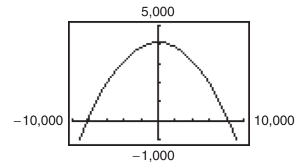
$$r = \frac{0.624}{1 + 0.847\sin(\pi/2)} \approx 0.338 \text{ astronomical units}$$

58. (a) $r = \dfrac{ep}{1 + e\sin\theta}$

Since the graph is a parabola, $e = 1$. The distance between the vertex and the focus (pole) is 4100, so the distance between the focus (pole) and the directrix is $p = 8200$.

$$r = \frac{8200}{1 + \sin\theta}$$

(b)



(c) When $\theta = 30°$, $r = \dfrac{8200}{1 + \sin 30°} \approx 5466.7$

Distance between surface of Earth and satellite:

$5466.7 - 4000 \approx 1467$ miles

(d) When $\theta = 60°$, $r = \dfrac{8200}{1 + \sin 60°} \approx 4394.4$

Distance between surface of Earth and satellite:

$4394.4 - 4000 \approx 394$ miles

59. True. The graphs represent the same hyperbola, although the graphs are not traced out in the same order as θ goes from 0 to 2π.

60. False. The graph has a horizontal directrix below the pole.

61. True. See Exercise 63.

$e = \frac{2}{3} < 1$

62. Answers will vary.

63.
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
$$\frac{r^2\cos^2\theta}{a^2} + \frac{r^2\sin^2\theta}{b^2} = 1$$
$$\frac{r^2\cos^2\theta}{a^2} + \frac{r^2(1 - \cos^2\theta)}{b^2} = 1$$
$$r^2b^2\cos^2\theta + r^2a^2 - r^2a^2\cos^2\theta = a^2b^2$$
$$r^2(b^2 - a^2)\cos^2\theta + r^2a^2 = a^2b^2$$

Since $b^2 - a^2 = -c^2$, we have:

$$-r^2c^2\cos^2\theta + r^2a^2 = a^2b^2$$
$$-r^2\left(\frac{c}{a}\right)^2\cos^2\theta + r^2 = b^2,\ e = \frac{c}{a}$$
$$-r^2e^2\cos^2\theta + r^2 = b^2$$
$$r^2(1 - e^2\cos^2\theta) = b^2$$
$$r^2 = \frac{b^2}{1 - e^2\cos^2\theta}$$

64.
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$
$$\frac{r^2\cos^2\theta}{a^2} - \frac{r^2\sin^2\theta}{b^2} = 1$$
$$\frac{r^2\cos^2\theta}{a^2} - \frac{r^2(1 - \cos^2\theta)}{b^2} = 1$$
$$r^2b^2\cos^2\theta - r^2a^2 + r^2a^2\cos^2\theta = a^2b^2$$
$$r^2(b^2 + a^2)\cos^2\theta - r^2a^2 = a^2b^2$$
$$a^2 + b^2 = c^2$$
$$r^2c^2\cos^2\theta - r^2a^2 = a^2b^2$$
$$r^2\left(\frac{c}{a}\right)^2\cos^2\theta - r^2 = b^2,\ e = \frac{c}{a}$$
$$r^2e^2\cos^2\theta - r^2 = b^2$$
$$r^2(e^2\cos^2\theta - 1) = b^2$$
$$r^2 = \frac{b^2}{e^2\cos^2\theta - 1}$$
$$= \frac{-b^2}{1 - e^2\cos^2\theta}$$

65. $\dfrac{x^2}{169} + \dfrac{y^2}{144} = 1$

$a = 13, b = 12, c = 5, e = \dfrac{5}{13}$

$r^2 = \dfrac{144}{1 - (25/169)\cos^2\theta} = \dfrac{24{,}336}{169 - 25\cos^2\theta}$

66. $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$

$a = 5, b = 4, c = 3, e = \dfrac{3}{5}$

$r^2 = \dfrac{400}{25 - 9\cos^2\theta}$

67. $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$

$a = 3, b = 4, c = 5, e = \dfrac{5}{3}$

$r^2 = \dfrac{-16}{1 - (25/9)\cos^2\theta} = \dfrac{144}{25\cos^2\theta - 9}$

68. $\dfrac{x^2}{36} - \dfrac{y^2}{4} = 1$

$a = 6, b = 2, c = 2\sqrt{10}, e = \dfrac{\sqrt{10}}{3}$

$r^2 = \dfrac{-4}{1 - (10/9)\cos^2\theta} = \dfrac{-36}{9 - 10\cos^2\theta}$

$= \dfrac{36}{10\cos^2\theta - 9}$

69. One focus: $(5, 0)$

Vertices: $(4, 0), (4, 0)$

$a = 4, c = 5 \implies b = 3$ and $e = \dfrac{5}{4}$

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

$r^2 = \dfrac{-9}{1 - (25/16)\cos^2\theta} = \dfrac{-144}{16 - 25\cos^2\theta}$

70. Ellipse

One focus: $(4, 0)$

Vertices: $(5, 0), (5, \pi)$

$a = 5, c = 4, b = 3, e = \dfrac{4}{5}$

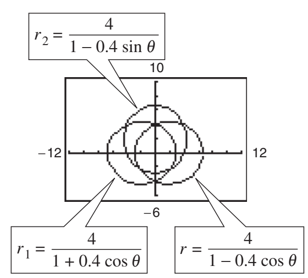
$r^2 = \dfrac{9}{1 - (16/25)\cos^2\theta} = \dfrac{225}{25 - 16\cos^2\theta}$

71. $r = \dfrac{4}{1 - 0.4\cos\theta}$

(a) Since $e < 1$, the conic is an ellipse.

(b) $r = \dfrac{4}{1 + 0.4\cos\theta}$ has a vertical directrix to the right of the pole and $r = \dfrac{4}{1 - 0.4\sin\theta}$ has a horizontal directrix below the pole. The given polar equation, $r = \dfrac{4}{1 - 0.4\cos\theta}$, has a vertical directrix to the left of the pole.

(c)



72. If e remains fixed and p changes, then the lengths of both the major axis and the minor axis change.

For example, graph $r = \dfrac{5}{1 - (2/3)\sin\theta}$, with $e = \dfrac{2}{3}$ and $p = \dfrac{15}{2}$, and

graph $r = \dfrac{6\frac{2}{3}}{1 - (2/3)\sin\theta}$, with $e = \dfrac{2}{3}$ and $p = 10$ on the same set of coordinate axes.

The first ellipse has a major axis of length 18 and a minor axis of length $6\sqrt{5}$, and the second ellipse has a major axis of length 21.6 and a minor axis of length $7.2\sqrt{5}$.

73.
$$\begin{aligned} 4\sqrt{3}\tan\theta - 3 &= 1 \\ 4\sqrt{3}\tan\theta &= 4 \\ \tan\theta &= \frac{1}{\sqrt{3}} \\ \theta &= \frac{\pi}{6} + n\pi \end{aligned}$$

74.
$$\begin{aligned} 6\cos x - 2 &= 1 \\ \cos x &= \frac{1}{2} \\ x &= \frac{\pi}{3} + 2\pi n, \frac{5\pi}{3} + 2\pi n \end{aligned}$$

75.
$$\begin{aligned} 12\sin^2\theta &= 9 \\ \sin^2\theta &= \frac{3}{4} \\ \sin\theta &= \pm\frac{\sqrt{3}}{2} \\ \theta &= \frac{\pi}{3} + n\pi, \frac{2\pi}{3} + n\pi \end{aligned}$$

76.
$$\begin{aligned} 9\csc^2 x - 10 &= 2 \\ \csc^2 x &= \frac{4}{3} \\ \sin x &= \pm\frac{\sqrt{3}}{2} \\ x &= \frac{\pi}{3} + \pi n, \frac{2\pi}{3} + \pi n \end{aligned}$$

77.
$$\begin{aligned} 2\cot x &= 5\cos\frac{\pi}{2} \\ 2\cot x &= 0 \\ \cot x &= 0 \\ x &= \frac{\pi}{2} + n\pi \end{aligned}$$

78.
$$\begin{aligned} \sqrt{2}\sec\theta &= 2\csc\frac{\pi}{4} \\ \cos\theta &= \frac{1}{2} \\ \theta &= \frac{\pi}{3} + 2\pi n, \frac{5\pi}{3} + 2\pi n \end{aligned}$$

For 79–82 use the following:

***u* and *v* are in Quadrant IV;** $\boldsymbol{\sin u = -\frac{3}{5} \Rightarrow \cos u = \frac{4}{5}; \quad \cos v = \frac{1}{\sqrt{2}} \Rightarrow \sin v = -\frac{1}{\sqrt{2}}}$

79.
$$\begin{aligned} \cos(u + v) &= \cos u\cos v - \sin u\sin v \\ &= \left(\frac{4}{5}\right)\left(\frac{1}{\sqrt{2}}\right) - \left(-\frac{3}{5}\right)\left(-\frac{1}{\sqrt{2}}\right) \\ &= \frac{4}{5\sqrt{2}} - \frac{3}{5\sqrt{2}} \\ &= \frac{1}{5\sqrt{2}} \\ &= \frac{\sqrt{2}}{10} \end{aligned}$$

80. $\sin u = -\dfrac{3}{5}, \cos u = \dfrac{4}{5}$

$\cos v = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}, \sin v = -\dfrac{\sqrt{2}}{2}$

$$\begin{aligned} \sin(u + v) &= \sin u\cos v + \cos u\sin v \\ &= \left(-\frac{3}{5}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{4}{5}\right)\left(-\frac{\sqrt{2}}{2}\right) \\ &= \frac{-7\sqrt{2}}{10} \end{aligned}$$

81.
$$\begin{aligned} \cos(u - v) &= \cos u\cos v + \sin u\sin v \\ &= \left(\frac{4}{5}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(-\frac{3}{5}\right)\left(-\frac{1}{\sqrt{2}}\right) \\ &= \frac{4}{5\sqrt{2}} + \frac{3}{5\sqrt{2}} \\ &= \frac{7}{5\sqrt{2}} \\ &= \frac{7\sqrt{2}}{10} \end{aligned}$$

82. $\sin u = -\dfrac{3}{5}, \cos u = \dfrac{4}{5}$

$\cos v = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}, \sin v = -\dfrac{\sqrt{2}}{2}$

$$\begin{aligned} \sin(u - v) &= \sin u\cos v - \cos u\sin v \\ &= \left(-\frac{3}{5}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{4}{5}\right)\left(-\frac{\sqrt{2}}{2}\right) \\ &= \frac{\sqrt{2}}{10} \end{aligned}$$

83. $\sin u = \frac{4}{5}, \ \frac{\pi}{2} < u < \pi \implies \cos u = -\frac{3}{5}$

$$\sin 2u = 2\sin u \cos u = 2\left(\frac{4}{5}\right)\left(-\frac{3}{5}\right) = -\frac{24}{25}$$

$$\cos 2u = \cos^2 u - \sin^2 u = \left(-\frac{3}{5}\right)^2 - \left(\frac{4}{5}\right)^2 = \frac{9}{25} - \frac{16}{25} = -\frac{7}{25}$$

$$\tan 2u = \frac{\sin 2u}{\cos 2u} = \frac{-24/25}{-7/25} = \frac{24}{7}$$

84. $\tan u = -\sqrt{3}, \frac{3\pi}{2} < u < 2\pi$

$\sin u = -\frac{\sqrt{3}}{2}, \cos u = \frac{1}{2}$

$$\sin 2u = 2\sin u \cos u = 2\left(-\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) = -\frac{\sqrt{3}}{2}$$

$$\cos 2u = \cos^2 u - \sin^2 u = \left(\frac{1}{2}\right)^2 - \left(-\frac{\sqrt{3}}{2}\right)^2 = -\frac{1}{2}$$

$$\tan 2u = \frac{2\tan u}{1 - \tan^2 u} = \frac{2(-\sqrt{3})}{1 - (-\sqrt{3})^2} = \sqrt{3}$$

85. $a_1 = 0, d = -\frac{1}{4}$

$$a_n = a_1 + (n-1)d = 0 + (n-1)\left(-\frac{1}{4}\right) = -\frac{1}{4}n + \frac{1}{4}$$

86. $a_n = a_1 + d(n-1)$

$a_n = 13 + 3(n-1)$

$a_n = 13 + 3n - 3$

$a_n = 10 + 3n$

87. $a_3 = 27, a_8 = 72$

$a_8 = a_3 + 5d$

$72 = 27 + 5d \implies d = 9$

$a_1 = 27 - 2(9) = 9$

$$a_n = a_1 + (n-1)d = 9 + (n-1)(9) = 9n$$

88. $a_n = a_k + d(n-k)$

$a_4 = a_1 + d(4-1)$

$9.5 = 5 + d(3)$

$d = 1.5$

$a_n = 5 + 1.5(n-1)$

89. $_{12}C_9 = \frac{12!}{(12-9)!9!} = \frac{12 \cdot 11 \cdot 10}{3!} = 220$

90. $_{18}C_{16} = \frac{18!}{16!(2!)} = \frac{18 \cdot 17}{2} = 153$

91. $_{10}P_3 = \frac{10!}{(10-3)!} = \frac{10!}{7!} = 10 \cdot 9 \cdot 8 = 720$

92. $_{29}P_2 = 29 \cdot 28 = 812$

Review Exercises for Chapter 10

1. Points: $(-1, 2)$ and $(2, 5)$

$m = \frac{5-2}{2-(-1)} = \frac{3}{3} = 1$

$\tan\theta = 1 \implies \theta = \frac{\pi}{4}$ radian $= 45°$

2. $m = \frac{4-7}{3-(-2)} = -\frac{3}{5} = \tan\theta$

$\theta = \pi + \arctan\left(-\frac{3}{5}\right)$

≈ 2.6012, or about $149.04°$

3. $y = 2x + 4 \implies m = 2$

$\tan\theta = 2 \implies \theta = \arctan 2 \approx 1.1071$ radians $\approx 63.43°$

4. $6x - 7y - 5 = 0$

$m = \frac{6}{7} = \tan\theta$

$\theta = \arctan\frac{6}{7} \approx 0.7086$, or about $40.60°$

5. $4x + y = 2 \implies y = -4x + 2 \implies m_1 = -4$

$-5x + y = -1 \implies y = 5x - 1 \implies m_2 = 5$

$$\tan\theta = \left|\frac{5 - (-4)}{1 + (-4)(5)}\right| = \frac{9}{19}$$

$$\theta = \arctan\frac{9}{19} \approx 0.4424 \text{ radian} \approx 25.35^\circ$$

6. $-5x + 3y = 3$

$-2x + 3y = 1$

$m_1 = \frac{5}{3}$

$m_2 = \frac{2}{3}$

$$\tan\theta = \left|\frac{(5/3) - (2/3)}{1 + (5/3)(2/3)}\right| = \frac{9}{19}$$

$\theta \approx 0.4424$, or about 25.35°

7. $2x - 7y = 8 \implies y = \frac{2}{7}x - \frac{8}{7} \implies m_1 = \frac{2}{7}$

$0.4x + y = 0 \implies y = -0.4x \implies m_2 = -0.4$

$$\tan\theta = \left|\frac{-0.4 - (2/7)}{1 + (2/7)(-0.4)}\right| = \frac{24}{31}$$

$$\theta = \arctan\left(\frac{24}{31}\right) \approx 0.6588 \text{ radian} \approx 37.75^\circ$$

8. $0.02x + 0.07y = 0.18$

$0.09x - 0.04y = 0.17$

$m_1 = -\frac{2}{7}$

$m_2 = \frac{9}{4}$

$$\tan\theta = \left|\frac{(9/4) - (-2/7)}{1 + (-2/7)(9/4)}\right| = \frac{71}{10}$$

$\theta \approx 1.4309$, or about 81.98°

9. $(1, 2) \implies x_1 = 1, y_1 = 2$

$x - y - 3 = 0 \implies A = 1, B = -1, C = -3$

$$d = \frac{|1(1) + (-1)(2) + (-3)|}{\sqrt{1^2 + (-1)^2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

10. $(0, 4) \implies x_1 = 0, y_1 = 4$

$x + 2y - 2 = 0 \implies A = 1, B = 2, C = -2$

$$d = \frac{|1(0) + (2)(4) + (-2)|}{\sqrt{1^2 + 2^2}} = \frac{6}{\sqrt{5}} = \frac{6\sqrt{5}}{5}$$

11. Hyperbola

12. A parabola is formed.

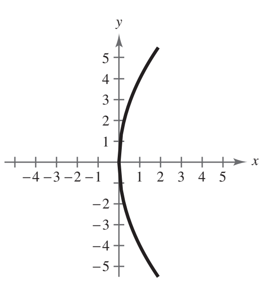
13. Vertex: $(0, 0) = (h, k)$

Focus: $(4, 0) \implies p = 4$

$(y - k)^2 = 4p(x - h)$

$(y - 0)^2 = 4(4)(x - 0)$

$y^2 = 16x$

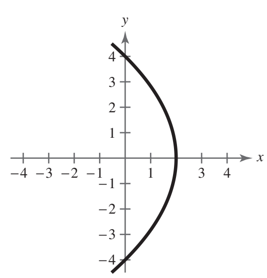


14. Vertex: $(2, 0) = (h, k)$

Focus: $(0, 0) \implies p = -2$

$(y - k)^2 = 4p(x - h)$

$y^2 = -8(x - 2)$

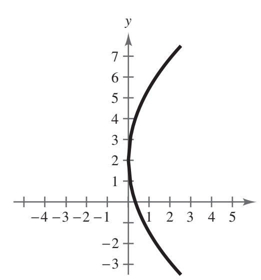


15. Vertex: $(0, 2) = (h, k)$

Directrix: $x = -3 \implies p = 3$

$(y - k)^2 = 4p(x - h)$

$(y - 2)^2 = 12x$

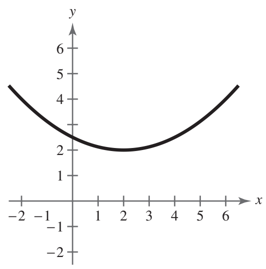


16. Vertex: $(2, 2) = (h, k)$

Directrix: $y = 0 \implies p = 2$

$(x - h)^2 = 4p(y - k)$

$(x - 2)^2 = 8(y - 2)$



17. $x^2 = -2y \implies p = -\dfrac{1}{2}$

Focus: $\left(0, -\dfrac{1}{2}\right)$

$d_1 = b + \dfrac{1}{2}$

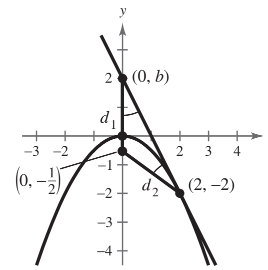
$d_2 = \sqrt{(2 - 0)^2 + \left(-2 + \dfrac{1}{2}\right)^2}$

$= \sqrt{4 + \dfrac{9}{4}} = \dfrac{5}{2}$

$d_1 = d_2$

$b + \dfrac{1}{2} = \dfrac{5}{2}$

$b = 2$



The slope of the line is

$m = \dfrac{-2 - 2}{2 - 0} = -2.$

Tangent line: $y = -2x + 2$

x-intercept: $(1, 0)$

18. $x^2 = -2y$

$p = -\dfrac{1}{2}$

Focus: $\left(0, -\dfrac{1}{2}\right)$

Tangent line through point $(-4, -8)$:

Slope: m

y-intercept: $(0, b)$

$d_1 = b + \dfrac{1}{2}$

$d_2 = \sqrt{(-4 - 0)^2 + \left(-8 + \dfrac{1}{2}\right)^2} = \dfrac{17}{2}$

$d_1 = d_2 \implies b = 8$

$m = \dfrac{-8 - 8}{-4 - 0} = 4$

$y = 4x + 8$

x-intercept of tangent line: $(-2, 0)$

19. Parabola

Opens downward

Vertex: $(0, 12)$

$(x - h)^2 = 4p(y - k)$

$x^2 = 4p(y - 12)$

Solution points: $(\pm 4, 10)$

$16 = 4p(10 - 12)$

$16 = -8p$

$-2 = p$

$x^2 = -8(y - 12)$

To find the x-intercepts, let $y = 0$.

$x^2 = 96$

$x = \pm\sqrt{96} = \pm 4\sqrt{6}$

At the base, the archway is $2\left(4\sqrt{6}\right) = 8\sqrt{6}$ meters wide.

20. $y^2 = 4px$

$p = 1.5$

$y^2 = 6x$

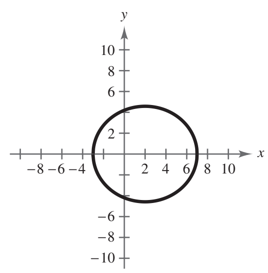
21. Vertices: $(-3, 0), (7, 0) \implies a = 5$
$(h, k) = (2, 0)$

Foci: $(0, 0), (4, 0) \implies c = 2$

$b^2 = a^2 - c^2 = 25 - 4 = 21$

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$$\frac{(x-2)^2}{25} + \frac{y^2}{21} = 1$$



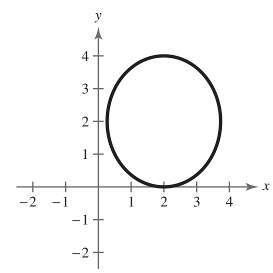
22. Vertices: $(2, 0), (2, 4) \implies a = 2, (h, k) = (2, 2)$

Foci: $(2, 1), (2, 3) \implies c = 1$

$b^2 = a^2 - c^2 = 4 - 1 = 3$

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

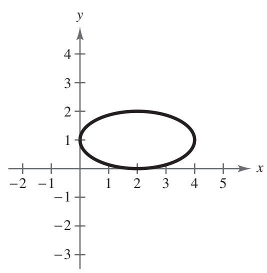
$$\frac{(x-2)^2}{3} + \frac{(y-2)^2}{4} = 1$$



23. Vertices: $(0, 1), (4, 1) \implies a = 2, (h, k) = (2, 1)$
Endpoints of minor axis: $(2, 0), (2, 2) \implies b = 1$

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

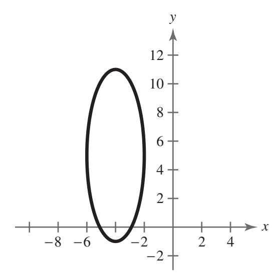
$$\frac{(x-2)^2}{4} + (y-1)^2 = 1$$



24. Vertices: $(-4, -1), (-4, 11) \implies a = 6, (h, k) = (-4, 5)$
Endpoints of the minor axis: $(-6, 5), (-2, 5) \implies b = 2$

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

$$\frac{(x+4)^2}{4} + \frac{(y-5)^2}{36} = 1$$



25. $2a = 10 \implies a = 5$

$b = 4$

$c^2 = a^2 - b^2 = 25 - 16 = 9 \implies c = 3$

The foci occur 3 feet from the center of the arch on a line connecting the tops of the pillars.

26. $\dfrac{x^2}{324} + \dfrac{y^2}{196} = 1$

$a = \sqrt{324} = 18, b = \sqrt{196} = 14$

$c = \sqrt{a^2 - b^2} = \sqrt{128} = 8\sqrt{2}$

Longest distance: $2a = 36$ feet

Shortest distance: $2b = 28$ feet

Distance between foci: $2c = 16\sqrt{2}$ feet

27. $\dfrac{(x+2)^2}{81} + \dfrac{(y-1)^2}{100} = 1$

$a = 10, b = 9, c = \sqrt{19}$

Center: $(-2, 1)$

Vertices: $(-2, 11)$ and $(-2, -9)$

Foci: $\left(-2, 1 \pm \sqrt{19}\right)$

Eccentricity: $e = \dfrac{\sqrt{19}}{10}$

28. $\dfrac{(x-5)^2}{1} + \dfrac{(y+3)^2}{36} = 1$

Center: $(5, -3)$

$a = 6, b = 1, c = \sqrt{a^2 - b^2} = \sqrt{35}$

Vertices: $(5, 3), (5, -9)$

Foci: $\left(5, -3 \pm \sqrt{35}\right)$

Eccentricity: $e = \dfrac{c}{a} = \dfrac{\sqrt{35}}{6}$

29. $16x^2 + 9y^2 - 32x + 72y + 16 = 0$

$$16(x^2 - 2x + 1) + 9(y^2 + 8y + 16) = -16 + 16 + 144$$

$$16(x - 1)^2 + 9(y + 4)^2 = 144$$

$$\frac{(x-1)^2}{9} + \frac{(y+4)^2}{16} = 1$$

$a = 4, b = 3, c = \sqrt{7}$

Center: $(1, -4)$

Vertices: $(1, 0)$ and $(1, -8)$

Foci: $\left(1, -4 \pm \sqrt{7}\right)$

Eccentricity: $e = \dfrac{\sqrt{7}}{4}$

30. $4x^2 + 25y^2 + 16x - 150y + 141 = 0$

$$4(x^2 + 4x + 4) + 25(y^2 - 6y + 9) = -141 + 16 + 225$$

$$\frac{(x+2)^2}{25} + \frac{(y-3)^2}{4} = 1$$

Center: $(-2, 3)$

$a = 5, b = 2, c = \sqrt{a^2 - b^2} = \sqrt{21}$

Vertices: $(3, 3), (-7, 3)$

Foci: $\left(-2 \pm \sqrt{21}, 3\right)$

Eccentricity: $e = \dfrac{c}{a} = \dfrac{\sqrt{21}}{5}$

31. Vertices: $(0, \pm 1) \implies a = 1, (h, k) = (0, 0)$

Foci: $(0, \pm 3) \implies c = 3$

$b^2 = c^2 - a^2 = 9 - 1 = 8$

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

$$y^2 - \frac{x^2}{8} = 1$$

32. Vertices: $(2, 2), (-2, 2) \implies a = 2, (h, k) = (0, 2)$

Foci: $(4, 2), (-4, 2) \implies c = 4$

$b^2 = c^2 - a^2 = 16 - 4 = 12$

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

$$\frac{x^2}{4} - \frac{(y-2)^2}{12} = 1$$

33. Foci: $(0, 0), (8, 0) \implies c = 4, (h, k) = (4, 0)$

Asymptotes: $y = \pm 2(x - 4) \implies \dfrac{b}{a} = 2, b = 2a$

$b^2 = c^2 - a^2 \implies 4a^2 = 16 - a^2 \implies$

$a^2 = \dfrac{16}{5}, b^2 = \dfrac{64}{5}$

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

$$\frac{(x-4)^2}{16/5} - \frac{y^2}{64/5} = 1$$

$$\frac{5(x-4)^2}{16} - \frac{5y^2}{64} = 1$$

34. Foci: $(3, \pm 2) \implies c = 2, (h, k) = (3, 0)$

Asymptotes: $y = \pm 2(x - 3) \implies \dfrac{a}{b} = 2, a = 2b$

$b^2 = c^2 - a^2 = 4 - 4b^2 \implies b^2 = \dfrac{4}{5}, a^2 = \dfrac{16}{5}$

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

$$\frac{y^2}{16/5} - \frac{(x-3)^2}{4/5} = 1 \implies \frac{5y^2}{16} - \frac{5(x-3)^2}{4} = 1$$

35. $\dfrac{(x-3)^2}{16} - \dfrac{(y+5)^2}{4} = 1$

$a = 4, b = 2, c = \sqrt{20} = 2\sqrt{5}$

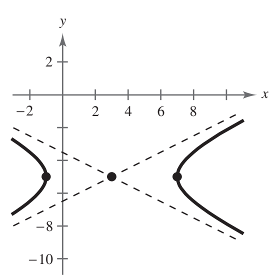
Center: $(3, -5)$

Vertices: $(7, -5)$ and $(-1, -5)$

Foci: $\left(3 \pm 2\sqrt{5}, -5\right)$

Asymptotes: $y = -5 \pm \dfrac{1}{2}(x - 3)$

$$y = \frac{1}{2}x - \frac{13}{2} \quad \text{or} \quad y = -\frac{1}{2}x - \frac{7}{2}$$



36. $\dfrac{(y-1)^2}{4} - x^2 = 1$

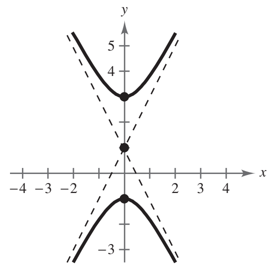
Center: $(0, 1)$

$a = 2, b = 1, c = \sqrt{a^2 + b^2} = \sqrt{5}$

Vertices: $(0, 3), (0, -1)$

Foci: $\left(0, 1 \pm \sqrt{5}\right)$

Asymptotes: $y = 1 \pm 2x$



37. $$9x^2 - 16y^2 - 18x - 32y - 151 = 0$$

$$9(x^2 - 2x + 1) - 16(y^2 + 2y + 1) = 151 + 9 - 16$$

$$9(x-1)^2 - 16(y+1)^2 = 144$$

$$\frac{(x-1)^2}{16} - \frac{(y+1)^2}{9} = 1$$

$a = 4, b = 3, c = 5$

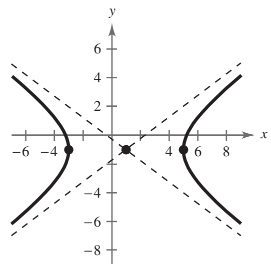
Center: $(1, -1)$

Vertices: $(5, -1)$ and $(-3, -1)$

Foci: $(6, -1)$ and $(-4, -1)$

Asymptotes: $y = -1 \pm \dfrac{3}{4}(x - 1)$

$$y = \frac{3}{4}x - \frac{7}{4} \quad \text{or} \quad y = -\frac{3}{4}x - \frac{1}{4}$$



38. $$-4x^2 + 25y^2 - 8x + 150y + 121 = 0$$

$$-4(x^2 + 2x + 1) + 25(y^2 + 6y + 9) = -121 - 4 + 225$$

$$\frac{(y+3)^2}{4} - \frac{(x+1)^2}{25} = 1$$

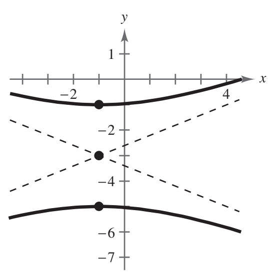
Center: $(-1, -3)$

$a = 2, b = 5, c = \sqrt{a^2 + b^2} = \sqrt{29}$

Vertices: $(-1, -1), (-1, -5)$

Foci: $\left(-1, -3 \pm \sqrt{29}\right)$

Asymptotes: $y = -3 \pm \dfrac{2}{5}(x + 1)$



39. Foci: $(\pm 100, 0) \implies c = 100$

Center: $(0, 0)$

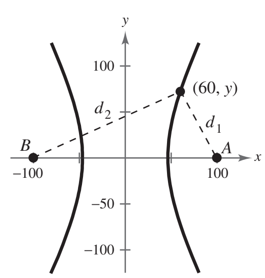
$\dfrac{d_2}{186{,}000} - \dfrac{d_1}{186{,}000} = 0.0005 \implies d_2 - d_1 = 93 = 2a \implies a = 46.5$

$b^2 = c^2 - a^2 = 100^2 - 46.5^2 = 7837.75$

$\dfrac{x^2}{2162.25} - \dfrac{y^2}{7837.75} = 1$

$y^2 = 7837.75\left(\dfrac{60^2}{2162.25} - 1\right) \approx 5211.5736$

$y \approx 72$ miles



40. $BD = AD + 6\left(\frac{1100}{5280}\right)$

$CD = AD + 8\left(\frac{1100}{5280}\right)$

$2a = CD - BD = 2\left(\frac{1100}{5280}\right)$

$a = \frac{5}{24}, c = 2 \implies b^2 = \frac{2279}{576}$

Thus, we have $\frac{576x^2}{25} - \frac{576y^2}{2279} = 1$ (x and y in miles) or $\frac{x^2}{1{,}210{,}000} - \frac{y^2}{110{,}303{,}600} = 1$ (x and y in feet).

OR:

$CD = AD + 8\left(\frac{1100}{5280}\right)$

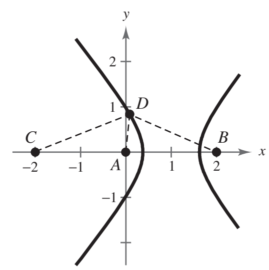
$BD = AD + 6\left(\frac{1100}{5280}\right)$

$2a = BD - AD = 6\left(\frac{1100}{5280}\right)$

$a = 3\left(\frac{5}{24}\right) = \frac{5}{8}, c = 1 \implies b^2 = \frac{39}{64}$

Center: $(1, 0)$

$\frac{64(x-1)^2}{25} - \frac{64y^2}{39} = 1$ (x and y in miles) or $\frac{(x-5280)^2}{10{,}890{,}000} - \frac{y^2}{16{,}988{,}400} = 1$ (x and y in feet).



41. $5x^2 - 2y^2 + 10x - 4y + 17 = 0$

$AC = 5(-2) = -10 < 0$

The graph is a hyperbola.

42. $-4y^2 + 5x + 3y + 7 = 0$

$AC = (0)(-4) = 0 \implies$ Parabola

43. $3x^2 + 2y^2 - 12x + 12y + 29 = 0$

$A = 3, C = 2$

$AC = 3(2) = 6 > 0$

The graph is an ellipse.

44. $4x^2 + 4y^2 - 4x + 8y - 11 = 0$

$A = 4, C = 4$

$A = C \implies$ Circle

45. $xy - 4 = 0$

$A = C = 0, B = 1$

$B^2 - 4AC = 1^2 - 4(0)(0) = 1 > 0$

The graph is a hyperbola.

$\cot 2\theta = 0 \implies 2\theta = \frac{\pi}{2} \implies \theta = \frac{\pi}{4}$

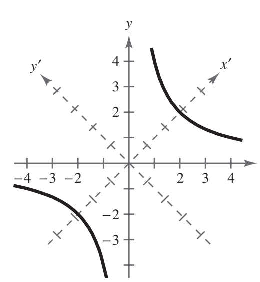
$x = x'\cos\frac{\pi}{4} - y'\sin\frac{\pi}{4} = \frac{x' - y'}{\sqrt{2}}$

$y = x'\sin\frac{\pi}{4} + y'\cos\frac{\pi}{4} = \frac{x' + y'}{\sqrt{2}}$

$\left(\frac{x' - y'}{\sqrt{2}}\right)\left(\frac{x' + y'}{\sqrt{2}}\right) - 4 = 0$

$\frac{(x')^2 - (y')^2}{2} = 4$

$\frac{(x')^2}{8} - \frac{(y')^2}{8} = 1$



46. $x^2 - 10xy + y^2 + 1 = 0$

$B^2 - 4AC = (-10)^2 - 4(1)(1) = 96 > 0 \implies$ Hyperbola

$$\cot 2\theta = \frac{A - C}{B} = \frac{1 - 1}{-10} = 0 \implies 2\theta = \frac{\pi}{2} \implies \theta = \frac{\pi}{4}$$

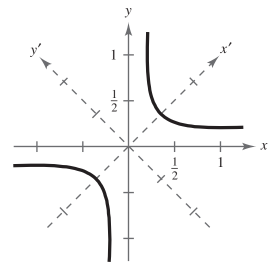
$$x = x'\cos\frac{\pi}{4} - y'\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}(x' - y')$$

$$y = x'\sin\frac{\pi}{4} + y'\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}(x' + y')$$

$$\frac{1}{2}(x' - y')^2 - 5(x' - y')(x' + y') + \frac{1}{2}(x' + y')^2 + 1 = 0$$

$$6(y')^2 - 4(x')^2 + 1 = 0$$

$$\frac{(x')^2}{1/4} - \frac{(y')^2}{1/6} = 1$$



47. $5x^2 - 2xy + 5y^2 - 12 = 0$

$A = C = 5, B = -2$

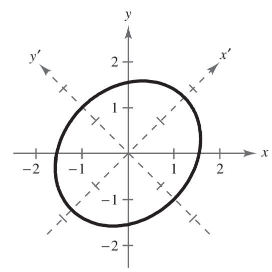
$B^2 - 4AC = (-2)^2 - 4(5)(5) = -96 < 0$

The graph is an ellipse.

$$\cot 2\theta = 0 \Rightarrow 2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$$

$$x = x'\cos\frac{\pi}{4} - y'\sin\frac{\pi}{4} = \frac{x' - y'}{\sqrt{2}}$$

$$y = x'\sin\frac{\pi}{4} + y'\cos\frac{\pi}{4} = \frac{x' + y'}{\sqrt{2}}$$



$$5\left(\frac{x' - y'}{\sqrt{2}}\right)^2 - 2\left(\frac{x' - y'}{\sqrt{2}}\right)\left(\frac{x' + y'}{\sqrt{2}}\right) + 5\left(\frac{x' + y'}{\sqrt{2}}\right)^2 - 12 = 0$$

$$\frac{5}{2}[(x')^2 - 2(x'y') + (y')^2] - [(x')^2 - (y')^2] + \frac{5}{2}[(x')^2 + 2(x'y') + (y')^2] = 12$$

$$4(x')^2 + 6(y')^2 = 12$$

$$\frac{(x')^2}{3} + \frac{(y')^2}{2} = 1$$

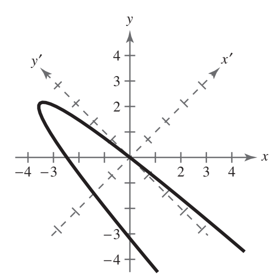
48. $4x^2 + 8xy + 4y^2 + 7\sqrt{2}x + 9\sqrt{2}y = 0$

$B^2 - 4AC = 8^2 - 4(4)(4) = 0 \implies$ Parabola

$$\cot 2\theta = \frac{A - C}{B} = \frac{4 - 4}{8} = 0 \implies 2\theta = \frac{\pi}{2} \implies \theta = \frac{\pi}{4}$$

$$x = x'\cos\frac{\pi}{4} - y'\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}(x' - y')$$

$$y = x'\sin\frac{\pi}{4} + y'\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}(x' + y')$$



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48. **—CONTINUED—**

$$\begin{aligned}2(x'-y')^2+4(x'-y')(x'+y')+2(x'+y')^2+7(x'-y')+9(x'+y')&=0\\ 8(x')^2+16x'+2y'&=0\\ y'&=-4(x')^2-8x'\\ y'&=-4((x')^2+2x'+1)+4\\ y'&=-4(x'+1)^2+4\\ y'-4&=-4(x'+1)^2\\ -4(x'+1)^2&=y'-4\\ y'&=-4(x'+1)^2+4\end{aligned}$$

49. (a) $16x^2-24xy+9y^2-30x-40y=0$

$B^2-4AC=(-24)^2-4(16)(9)=0$

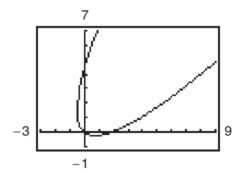
The graph is a parabola.

(b) To use a graphing utility, we need to solve for y in terms of x.

$9y^2+(-24x-40)y+(16x^2-30x)=0$

$$y=\frac{-(-24x-40)\pm\sqrt{(-24x-40)^2-4(9)(16x^2-30x)}}{2(9)}$$

$$=\frac{(24x+40)\pm\sqrt{(24x+40)^2-36(16x^2-30x)}}{18}$$

(c) 

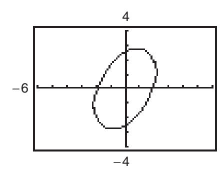
50. (a) $13x^2-8xy+7y^2-45=0$

$B^2-4AC=(-8)^2-4(13)(7)=-300<0\implies$ Ellipse

(b) Use the Quadratic Formula to solve for y in terms of x:

$7y^2-8xy+13x^2-45=0$

$$y=\frac{1}{14}\left[8x\pm\sqrt{64x^2-28(13x^2-45)}\right]$$

(c) 

51. (a) $x^2+y^2+2xy+2\sqrt{2}x-2\sqrt{2}y+2=0$

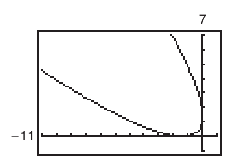
$B^2-4AC=2^2-4(1)(1)=0$

The graph is a parabola.

(b) To use a graphing utility, we need to solve for y in terms of x.

$y^2+\left(2x-2\sqrt{2}\right)y+\left(x^2+2\sqrt{2}x+2\right)=0$

$$y=\frac{-\left(2x-2\sqrt{2}\right)\pm\sqrt{\left(2x-2\sqrt{2}\right)^2-4\left(x^2+2\sqrt{2}x+2\right)}}{2}$$

(c) 

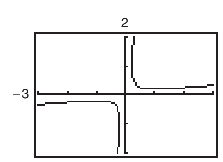
52. (a) $x^2-10xy+y^2+1=0$

Since $B^2-4AC=(-10)^2-4(1)(1)>0\implies$ Hyperbola

(b) Use the Quadratic Formula to solve for y in terms of x:

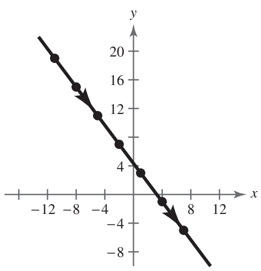
$y^2-10xy+x^2+1=0$

$$y=\tfrac{1}{2}\left[10x\pm\sqrt{100x^2-4(x^2+1)}\right]$$

(c) 

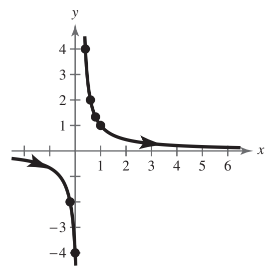
53. $x = 3t - 2, y = 7 - 4t$

t	-3	-2	0	1	2	3
x	-11	-8	-2	1	4	7
y	19	15	7	3	-1	-5

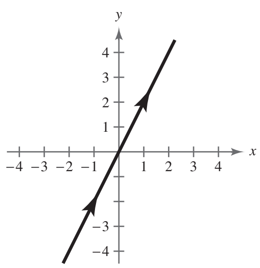


54. $x = \frac{1}{5}t$ and $y = \frac{4}{t-1}$

t	-1	0	2	3	4	5
x	$-\frac{1}{5}$	0	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	1
y	-2	-4	4	2	$\frac{4}{3}$	1



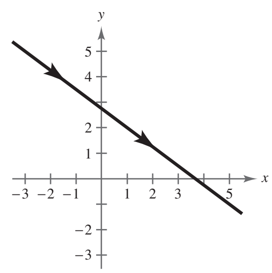
55. (a)



(b) $x = 2t \implies \frac{x}{2} = t$

$y = 4t \implies y = 4\left(\frac{x}{2}\right) = 2x$

56. (a)



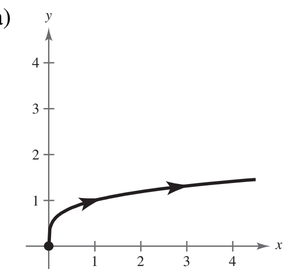
(b) $x = 1 + 4t, y = 2 - 3t$

$t = \frac{x-1}{4}$

$y = 2 - 3\left(\frac{x-1}{4}\right)$

$3x + 4y = 11$

57. (a)



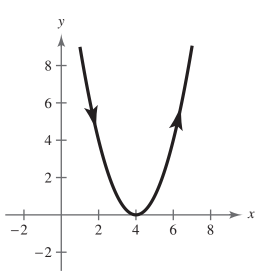
(b) $x = t^2,\ x \geq 0$

$y = \sqrt{t} \implies y^2 = t$

$x = (y^2)^2 \implies x$

$= y^4 \implies y = \sqrt[4]{x}$

58. (a)

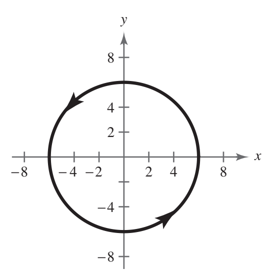


(b) $x = t + 4, y = t^2$

$t = x - 4$

$y = (x - 4)^2$

59. (a)



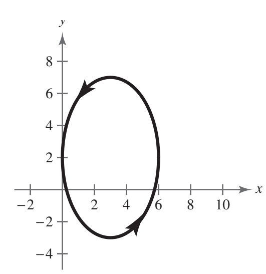
(b) $x = 6\cos\theta, y = 6\sin\theta$

$\cos\theta = \frac{x}{6}, \sin\theta = \frac{y}{6}$

$\frac{x^2}{36} + \frac{y^2}{36} = 1$

$x^2 + y^2 = 36$

60. (a)



(b) $x = 3 + 3\cos\theta, y = 2 + 5\sin\theta$

$\cos\theta = \frac{x-3}{3}, \sin\theta = \frac{y-2}{5}$

$\frac{(x-3)^2}{9} + \frac{(y-2)^2}{25} = 1$

61. Center: $(5, 4)$

Radius: 6

$x = h + r\cos\theta = 5 + 6\cos\theta$

$y = k + r\sin\theta = 4 + 6\sin\theta$

62. $(h, k) = (-3, 4)$

$2a = 8 \implies a = 4$

$2b = 6 \implies b = 3$

$$\frac{(x+3)^2}{16} + \frac{(y-4)^2}{9} = 1$$

$x = -3 + 4\cos\theta$

$y = 4 + 3\sin\theta$

This solution is not unique.

63. Hyperbola

Vertices: $(0, \pm 4)$

Foci: $(0, \pm 5)$

Center: $(0, 0)$

$a = 4, c = 5, b = \sqrt{c^2 - a^2} = 3$

$x = 3\tan\theta, y = 4\sec\theta$

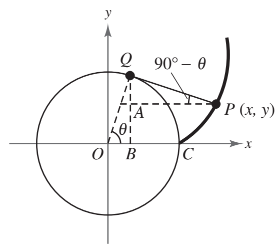
64. $y = \overline{QB} - \overline{QA}$

$\overline{QP} = \text{arc } QC = r\theta$

$\overline{QA} = r\theta\sin(90° - \theta)$

$= r\theta\cos\theta$

$\overline{QB} = r\sin\theta$



Therefore,

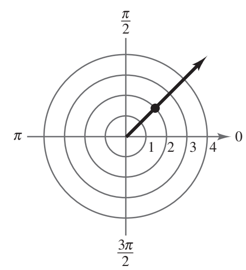
$y = r\sin\theta - r\theta\cos\theta = r(\sin\theta - \theta\cos\theta)$.

Similarly, $x = \overline{OB} + \overline{AP}$.

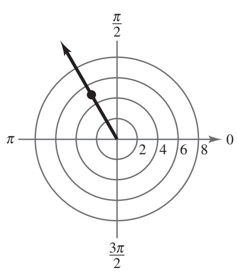
Therefore, $x = r\cos\theta + r\theta\sin\theta = r(\cos\theta + \theta\sin\theta)$.

65. Polar coordinates: $\left(2, \dfrac{\pi}{4}\right)$

Additional polar representations: $\left(2, -\dfrac{7\pi}{4}\right), \left(-2, \dfrac{5\pi}{4}\right)$

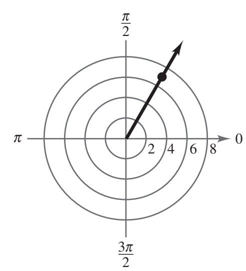


66. Polar coordinates: $\left(-5, -\dfrac{\pi}{3}\right) = \left(5, \dfrac{2\pi}{3}\right)$ or $\left(-5, \dfrac{5\pi}{3}\right)$



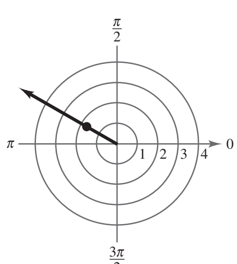
67. Polar coordinates: $(-7, 4.19)$

Additional polar representations: $(7, 1.05), (-7, -2.09)$



68. Polar coordinates:

$\left(\sqrt{3}, 2.62\right) \approx \left(\sqrt{3}, -3.66\right)$ or $\left(-\sqrt{3}, 5.76\right)$



69. Polar coordinates: $\left(-1, \dfrac{\pi}{3}\right)$

$x = -1\cos\dfrac{\pi}{3} = -\dfrac{1}{2}$

$y = -1\sin\dfrac{\pi}{3} = -\dfrac{\sqrt{3}}{2}$

Rectangular coordinates: $\left(-\dfrac{1}{2}, -\dfrac{\sqrt{3}}{2}\right)$

70. Polar coordinates: $\left(2, \dfrac{5\pi}{4}\right) = (r, \theta)$

$x = r\cos\theta = 2\cos\dfrac{5\pi}{4} = -\sqrt{2}$

$y = r\sin\theta = 2\sin\dfrac{5\pi}{4} = -\sqrt{2}$

Rectangular coordinates: $\left(-\sqrt{2}, -\sqrt{2}\right)$

71. Polar coordinates: $\left(3, \dfrac{3\pi}{4}\right)$

$x = 3\cos\dfrac{3\pi}{4} = -\dfrac{3\sqrt{2}}{2}$

$y = 3\sin\dfrac{3\pi}{4} = \dfrac{3\sqrt{2}}{2}$

Rectangular coordinates: $\left(-\dfrac{3\sqrt{2}}{2}, \dfrac{3\sqrt{2}}{2}\right)$

72. Polar coordinates: $\left(0, \dfrac{\pi}{2}\right) = (r, \theta)$

$x = r\cos\theta = 0\cos\dfrac{\pi}{2} = 0$

$y = r\sin\theta = 0\sin\dfrac{\pi}{2} = 0$

Rectangular coordinates: $(0, 0)$

73. Rectangular coordinates: $(0, 2)$

$r = \pm\sqrt{0^2 + 2^2} = \pm 2$

$\tan\theta$ is undefined $\implies \theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

Polar coordinates: $\left(2, \dfrac{\pi}{2}\right)$ or $\left(-2, \dfrac{3\pi}{2}\right)$

74. Rectangular coordinates: $\left(-\sqrt{5}, \sqrt{5}\right)$

Polar coordinates:

$r = \sqrt{\left(-\sqrt{5}\right)^2 + \left(\sqrt{5}\right)^2} = \sqrt{10}$

$\tan\theta = -1, \theta = \dfrac{3\pi}{4}$

$\left(\sqrt{10}, \dfrac{3\pi}{4}\right)$

75. Rectangular coordinates: $(4, 6)$

$r = \pm\sqrt{4^2 + 6^2} = \pm\sqrt{52} = \pm 2\sqrt{13}$

$\tan\theta = \dfrac{6}{4} \implies \theta \approx 0.9828, 4.1244$

Polar coordinates: $\left(2\sqrt{13}, 0.9828\right)$ or $\left(-2\sqrt{13}, 4.1244\right)$

76. Rectangular coordinates: $(3, -4)$

Polar coordinates:

$r = \sqrt{3^2 + (-4)^2} = 5$

$\tan\theta = -\dfrac{4}{3}, \theta \approx -0.9273$

$(5, 5.356)$

77.
$$\begin{aligned} x^2 + y^2 &= 49 \\ r^2 &= 49 \\ r &= 7 \end{aligned}$$

78.
$$\begin{aligned} x^2 + y^2 &= 20 \\ x^2 + y^2 &= r^2 \\ r^2 &= 20 \\ r &= 2\sqrt{5} \end{aligned}$$

79.
$$\begin{aligned} x^2 + y^2 - 6y &= 0 \\ r^2 - 6r\sin\theta &= 0 \\ r(r - 6\sin\theta) &= 0 \\ r = 0 \text{ or } r &= 6\sin\theta \end{aligned}$$

Since $r = 6\sin\theta$ contains $r = 0$, we just have $r = 6\sin\theta$.

80.
$$\begin{aligned} x^2 + y^2 - 4x &= 0 \\ r^2 - 4r\cos\theta &= 0 \\ r &= 4\cos\theta \end{aligned}$$

81.
$$\begin{aligned} xy &= 5 \\ (r\cos\theta)(r\sin\theta) &= 5 \\ r^2 &= \frac{5}{\sin\theta\cos\theta} \\ &= \frac{10}{\sin 2\theta} = 10\csc 2\theta \end{aligned}$$

82.
$$\begin{aligned} xy &= -2 \\ r\cos\theta\, r\sin\theta &= -2 \\ r^2\cos\theta\,\sin\theta &= -2 \\ r^2 &= \frac{-2}{\cos\theta\sin\theta} \\ r^2 &= -2\sec\theta\,\csc\theta \\ r^2 &= -4\csc 2\theta \end{aligned}$$

83.
$$\begin{aligned} r &= 5 \\ r^2 &= 25 \\ x^2 + y^2 &= 25 \end{aligned}$$

84.
$$\begin{aligned} r &= 12 \\ r^2 &= 144 \\ &= x^2 + y^2 \text{ or } x^2 + y^2 = 144 \end{aligned}$$

85.
$$r = 3\cos\theta$$
$$r^2 = 3r\cos\theta$$
$$x^2 + y^2 = 3x$$

86. Because $y = r\sin\theta$ and r is given as $8\sin\theta$, $y = 8\sin\theta\sin\theta = 8\sin^2\theta$.
$$r = 8\sin\theta$$
$$r^2 = 64\sin^2\theta$$
$$r^2 = 8(8\sin^2\theta)$$
$$x^2 + y^2 = 8y$$
$$x^2 + y^2 - 8y = 0$$

87.
$$r^2 = \sin\theta$$
$$r^3 = r\sin\theta$$
$$\left(\pm\sqrt{x^2 + y^2}\right)^3 = y$$
$$(x^2 + y^2)^3 = y^2$$
$$x^2 + y^2 = y^{2/3}$$

88.
$$r^2 = \cos 2\theta$$
$$r^2 = \left(\frac{x}{r}\right)^2 - \left(\frac{y}{r}\right)^2$$
$$(x^2 + y^2)^2 = x^2 - y^2$$

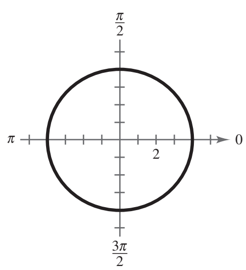
89. $r = 4$

Circle of radius 4 centered at the pole

Symmetric with respect to $\theta = \pi/2$, the polar axis and the pole

Maximum value of $|r| = 4$, for all values of θ

Zeros: None

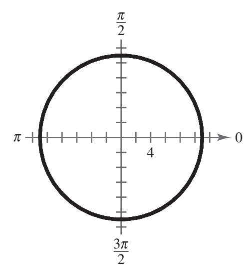


90. $r = 11$

Symmetry: $\theta = \dfrac{\pi}{2}$, polar axis, pole

Maximum value of $|r|$: 11, for all values of θ

Zeros of r: none



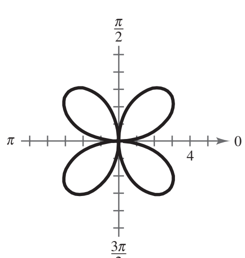
91. $r = 4\sin 2\theta$

Rose curve $(n = 2)$ with 4 petals

Symmetric with respect to $\theta = \pi/2$, the polar axis, and the pole

Maximum value of $|r| = 4$ when $\theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$

Zeros: $r = 0$ when $\theta = 0, \dfrac{\pi}{2}, \pi, \dfrac{3\pi}{2}$



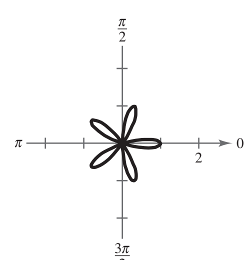
92. $r = \cos 5\theta$

Symmetry: polar axis

Maximum value of $|r|$: $|r| = 1$ when

$\theta = 0, \dfrac{2\pi}{5}, \dfrac{4\pi}{5}, \dfrac{6\pi}{5}, \dfrac{8\pi}{5}$

Zeros of r: $r = 0$ when $\theta = \dfrac{\pi}{10}, \dfrac{3\pi}{10}, \dfrac{\pi}{2}, \dfrac{7\pi}{10}, \dfrac{9\pi}{10}$



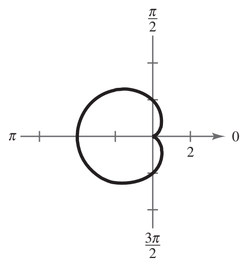
93. $r = -2(1 + \cos\theta)$

Symmetric with respect to the polar axis

Maximum value of $|r| = 4$ when $\theta = 0$

Zeros: $r = 0$ when $\theta = \pi$

$\dfrac{a}{b} = \dfrac{2}{2} = 1 \implies$ Cardioid

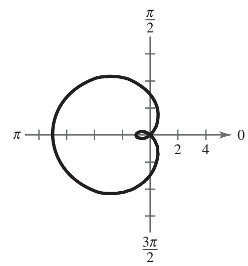


94. $r = 3 - 4\cos\theta$

Symmetry: polar axis

Maximum value of $|r|$: $|r| = 7$ when $\theta = \pi$

Zeros of r: $r = 0$ when $\theta = \arccos\dfrac{3}{4}, 2\pi - \arccos\dfrac{3}{4}$



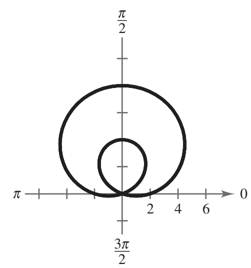
95. $r = 2 + 6\sin\theta$

Limaçon with inner loop

$r = f(\sin\theta) \implies \theta = \dfrac{\pi}{2}$ symmetry

Maximum value: $|r| = 8$ when $\theta = \dfrac{\pi}{2}$

Zeros: $2 + 6\sin\theta = 0 \implies \sin\theta = -\dfrac{1}{3} \implies \theta \approx 3.4814, 5.9433$



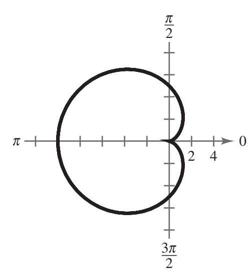
96. $r = 5 - 5\cos\theta$

$r = 5(1 - \cos\theta)$

Symmetry: polar axis

Maximum values of $|r|$: $|r| = 10$ when $\theta = \pi$

Zeros of r: $r = 0$ when $\theta = 0, 2\pi$



97. $r = -3\cos 2\theta$

Rose curve with 4 petals

$r = f(\cos\theta) \implies$ polar axis symmetry

$\theta = \dfrac{\pi}{2}$: $r = -3\cos 2(\pi - \theta) = -3\cos(2\pi - 2\theta) = -3\cos 2\theta$

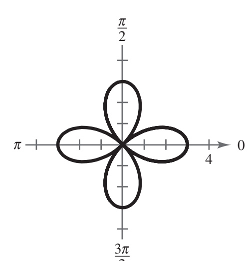
Equivalent equation $\implies \theta = \dfrac{\pi}{2}$ symmetry

Pole: $r = -3\cos 2(\pi + \theta) = -3\cos(2\pi + 2\theta) = -3\cos 2\theta$

Equivalent equation $\implies$ pole symmetry

Maximum value: $|r| = 3$ when $\theta = 0, \dfrac{\pi}{2}, \pi, \dfrac{3\pi}{2}$

Zeros: $-3\cos 2\theta = 0$ when $\cos 2\theta = 0 \implies \theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$

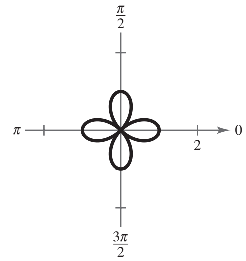


98. $r = \cos 2\theta$

Symmetry: polar axis

Maximum value of $|r|$: $|r| = 1$ when $\theta = 0, \dfrac{\pi}{2}, \pi, \dfrac{3\pi}{2}$

Zeros of r: $r = 0$ when $\theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$

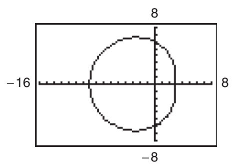


99. $r = 3(2 - \cos\theta)$

$= 6 - 3\cos\theta$

$\dfrac{a}{b} = \dfrac{6}{3} = 2$

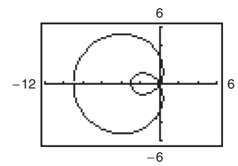
The graph is a convex limaçon.



100. $r = 3(1 - 2\cos\theta)$

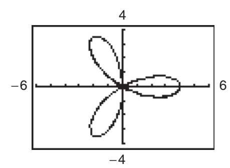
$r = 3 - 6\cos\theta$

Limaçon with inner loop.



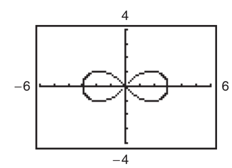
101. $r = 4\cos 3\theta$

The graph is a rose curve with 3 petals.



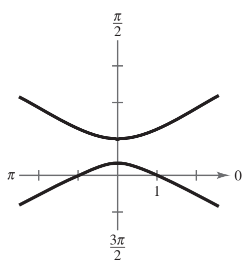
102. $r^2 = 9\cos 2\theta$

Lemniscate



103. $r = \dfrac{1}{1 + 2\sin\theta}, e = 2$

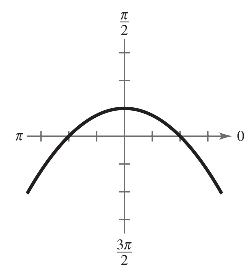
Hyperbola symmetric with respect to $\theta = \dfrac{\pi}{2}$ and having vertices at $\left(\dfrac{1}{3}, \dfrac{\pi}{2}\right)$ and $\left(-1, \dfrac{3\pi}{2}\right)$.



104. $r = \dfrac{2}{1 + \sin\theta}$

$e = 1 \implies$ parabola

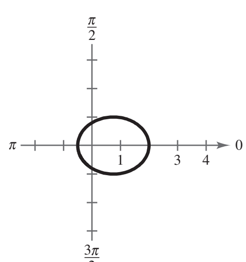
Vertex: $\left(1, \dfrac{\pi}{2}\right)$



105. $r = \dfrac{4}{5 - 3\cos\theta}$

$r = \dfrac{4/5}{1 - (3/5)\cos\theta}, e = \dfrac{3}{5}$

Ellipse symmetric with respect to the polar axis and having vertices at $(2, 0)$ and $(1/2, \pi)$.

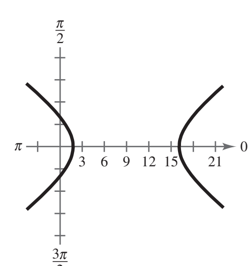


106. $r = \dfrac{16}{4 + 5\cos\theta}$

$r = \dfrac{4}{1 + (5/4)\cos\theta}$

$e = \dfrac{5}{4} > 1 \implies$ Hyperbola

Vertices: $\left(\dfrac{16}{9}, 0\right), (-16, \pi)$



107. Parabola: $r = \dfrac{ep}{1 - e\cos\theta}$, $e = 1$

Vertex: $(2, \pi)$

Focus: $(0, 0) \implies p = 4$

$$r = \frac{4}{1 - \cos\theta}$$

108. Parabola: $r = \dfrac{ep}{1 + e\sin\theta}$, $e = 1$

Vertex: $\left(2, \dfrac{\pi}{2}\right)$

Focus: $(0, 0) \implies p = 4$

$$r = \frac{4}{1 + \sin\theta}$$

109. Ellipse: $r = \dfrac{ep}{1 - e\cos\theta}$

Vertices: $(5, 0), (1, \pi) \implies a = 3$

One focus: $(0, 0) \implies c = 2$

$$e = \frac{c}{a} = \frac{2}{3}, p = \frac{5}{2}$$

$$r = \frac{(2/3)(5/2)}{1 - (2/3)\cos\theta} = \frac{5/3}{1 - (2/3)\cos\theta}$$

$$= \frac{5}{3 - 2\cos\theta}$$

110. Hyperbola: $r = \dfrac{ep}{1 + e\cos\theta}$

Vertices: $(1, 0), (7, 0) \implies a = 3$

One focus: $(0, 0) \implies c = 4$

$$e = \frac{c}{a} = \frac{4}{3}, p = \frac{7}{4}$$

$$r = \frac{(4/3)(7/4)}{1 + (4/3)\cos\theta} = \frac{7/3}{1 + (4/3)\cos\theta} = \frac{7}{3 + 4\cos\theta}$$

111.
$$\begin{aligned} a + c &= 122{,}800 + 4000 &\implies\quad a + c &= 126{,}800 \\ a - c &= 119 + 4000 &\implies\quad \underline{a - c} &\underline{= 4{,}119} \\ & & 2a &= 130{,}919 \\ & & a &= 65{,}459.5 \\ & & c &= 61{,}340.5 \end{aligned}$$

$$e = \frac{c}{a} = \frac{61{,}340.5}{65{,}459.5} \approx 0.937$$

$$r = \frac{ep}{1 - e\cos\theta} \approx \frac{0.937p}{1 - 0.937\cos\theta}$$

$r = 126{,}800$ when $\theta = 0$

$$126{,}800 = \frac{ep}{1 - e\cos 0}$$

$$ep = 126{,}800\left(1 - \frac{61{,}340.5}{65{,}459.5}\right) \approx 7978.81$$

Thus, $r \approx \dfrac{7978.81}{1 - 0.937\cos\theta}$.

When $\theta = \dfrac{\pi}{3}$, $r \approx \dfrac{7978.81}{1 - 0.937\cos(\pi/3)} \approx 15{,}011.87$ miles.

The distance from the surface of Earth and the satellite is $15{,}011.87 - 4000 \approx 11{,}011.87$ miles.

112. Parabola: $r = \dfrac{ep}{1 + e\sin\theta}$, $e = 1$

Vertex: $\left(6{,}000{,}000, \dfrac{\pi}{2}\right)$

Focus: $(0, 0) \implies p = 12{,}000{,}000$

$$r = \frac{12{,}000{,}000}{1 + \sin\theta}$$

$$\theta = -\frac{\pi}{3}$$

$r \approx 89{,}600{,}000$ miles

113. False. When classifying equations of the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, its graph can be determined by its discriminant. For a graph to be a parabola, its discriminant, $B^2 - 4AC$, must equal zero. So, if $B = 0$, then A **or** C equals 0, but not both.

114. False.

$\dfrac{x^2}{4} - y^4 = 1$ is a fourth-degree equation.

The equation of a hyperbola is a second degree equation.

115. False. The following are **two** sets of parametric equations for the line.

$x = t, y = 3 - 2t$

$x = 3t, y = 3 - 6t$

116. False.

$(r, \theta), (r, \theta + 2\pi), (-r, \theta + \pi)$, etc.

All represent the same point.

117. $2a = 10 \implies a = 5$

b must be less than 5; $0 < b < 5$.

As b approaches 5, the ellipse becomes more circular and approaches a circle of radius 5.

118. The orientation would be reversed.

119. $x = 4 \cos t$ and $y = 3 \sin t$

(a) $x = 4 \cos 2t$ and $y = 3 \sin 2t$

The speed would double.

(b) $x = 5 \cos t$ and $y = 3 \sin t$

The elliptical orbit would be flatter. The length of the major axis is greater.

120. (a) $\left(-4, \frac{\pi}{6}\right), \left(4, \frac{\pi}{6}\right)$: symmetric about the pole

(b) $\left(4, -\frac{\pi}{6}\right), \left(4, \frac{\pi}{6}\right)$: symmetric about the polar axis

(c) $\left(-4, -\frac{\pi}{6}\right), \left(4, \frac{\pi}{6}\right)$: symmetric about the $\theta = \frac{\pi}{2}$ axis.

121. (a) $x^2 + y^2 = 25$

$r = 5$

The graphs are the same. They are both circles centered at $(0, 0)$ with a radius of 5.

(b) $x - y = 0 \implies y = x$

$\theta = \frac{\pi}{4}$

The graphs are the same. They are both lines with slope 1 and intercept $(0, 0)$.

122. Area of the circle: $A = 100\pi$

Area of the ellipse: $A = \pi ab = \pi a(10) = 2(100\pi) \implies a = 20$

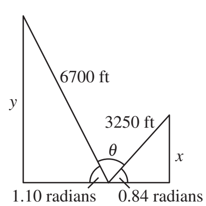
Length of major axis: $2a = 40$

Problem Solving for Chapter 10

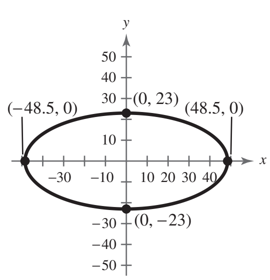
1. (a) $\theta = \pi - 1.10 - 0.84 \approx 1.2016$ radians

(b) $\sin 0.84 = \dfrac{x}{3250} \implies x = 3250 \sin 0.84 \approx 2420$ feet

$\sin 1.10 = \dfrac{y}{6700} \implies y = 6700 \sin 1.10 \approx 5971$ feet



2.



(a) Let $(0, 0)$ represent the center of the ellipse. Then $2a = 97 \implies a = 48.5$ and $2b = 46 \implies b = 23$.

$$\frac{x^2}{(48.5)^2} + \frac{y^2}{23^2} = 1$$

$$\frac{x^2}{2352.25} + \frac{y^2}{529} = 1$$

(b) $c^2 = a^2 - b^2 = 2352.25 - 529 = 1823.25$

$c \approx 42.7$

The foci are $2c \approx 85.4$ feet apart.

(c) $A = \pi ab = \pi(48.5)(23) = 1115.5\pi \approx 3504.45$ square feet

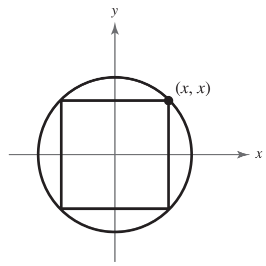
3. Since the axis of symmetry is the x-axis, the vertex is $(h, 0)$ and $y^2 = 4p(x - h)$. Also, since the focus is $(0, 0)$, $0 - h = p \Rightarrow h = -p$ and $y^2 = 4p(x + p)$.

4. Let (x, x) be the corner of the square in Quadrant I.

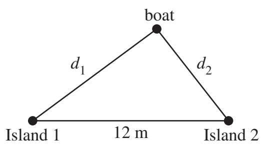
$A = 4x^2$

$\frac{x^2}{a^2} + \frac{x^2}{b^2} = 1 \Rightarrow x^2 = \frac{a^2b^2}{a^2 + b^2}$

Thus, $A = \frac{4a^2b^2}{a^2 + b^2}$.

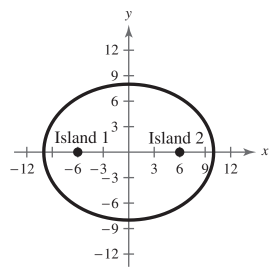


5. (a)



Since $d_1 + d_2 \le 20$, by definition, the outer bound that the boat can travel is an ellipse.The islands are the foci.

(b)



Island 1 is located at $(-6, 0)$ and Island 2 is located at $(6, 0)$.

(c) $d_1 + d_2 = 2a = 20 \Rightarrow a = 10$

The boat traveled 20 miles. The vertex is $(10, 0)$.

(d) $c = 6, a = 10 \Rightarrow b^2 = a^2 - c^2 = 64$

$$\frac{x^2}{100} + \frac{y^2}{64} = 1$$

6. Foci: $(2, 2)$ and $(10, 2)$ $\Rightarrow$ Center is $(6, 2)$ and $c = 4$

$|d_2 - d_1| = 2a = 6 \Rightarrow a = 3$

$c^2 = a^2 + b^2 \Rightarrow 16 = 9 + b^2 \Rightarrow b^2 = 7$

Horizontal transverse axis

$$\frac{(x - 6)^2}{9} - \frac{(y - 2)^2}{7} = 1$$

7. $Ax^2 + Cy^2 + Dx + Ey + F = 0$

Assume that the conic is *not* degenerate.

(a) $A = C, A \neq 0$

$$Ax^2 + Ay^2 + Dx + Ey + F = 0$$

$$x^2 + y^2 + \frac{D}{A}x + \frac{E}{A}y + \frac{F}{A} = 0$$

$$\left(x^2 + \frac{D}{A}x + \frac{D^2}{4A^2}\right) + \left(y^2 + \frac{E}{A}y + \frac{E^2}{4A^2}\right) = -\frac{F}{A} + \frac{D^2}{4A^2} + \frac{E^2}{4A^2}$$

$$\left(x + \frac{D}{2A}\right)^2 + \left(y + \frac{E}{2A}\right)^2 = \frac{D^2 + E^2 - 4AF}{4A^2}$$

This is a circle with center $\left(-\frac{D}{2A}, -\frac{E}{2A}\right)$ and radius $\frac{\sqrt{D^2 + E^2 - 4AF}}{2|A|}$.

(b) $A = 0$ or $C = 0$ (but not both). Let $C = 0$.

$$Ax^2 + Dx + Ey + F = 0$$

$$x^2 + \frac{D}{A}x = -\frac{E}{A}y - \frac{F}{A}$$

$$x^2 + \frac{D}{A}x + \frac{D^2}{4A^2} = -\frac{E}{A}y - \frac{F}{A} + \frac{D^2}{4A^2}$$

$$\left(x + \frac{D}{2A}\right)^2 = -\frac{E}{A}\left(y + \frac{F}{E} - \frac{D^2}{4AE}\right)$$

This is a parabola with vertex $\left(-\frac{D}{2A}, \frac{D^2 - 4AF}{4AE}\right)$.

$A = 0$ yields a similar result.

—CONTINUED—

7. —CONTINUED—

(c) $AC > 0 \implies A$ and C are either both positive or are both negative (if that is the case, move the terms to the other side of the equation so that they are both positive).

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

$$A\left(x^2 + \frac{D}{A}x + \frac{D^2}{4A^2}\right) + C\left(y^2 + \frac{E}{C}y + \frac{E^2}{4C^2}\right) = -F + \frac{D^2}{4A} + \frac{E^2}{4C}$$

$$A\left(x + \frac{D}{2A}\right)^2 + C\left(y + \frac{E}{2C}\right)^2 = \frac{CD^2 + AE^2 - 4ACF}{4AC}$$

$$\frac{\left(x + \frac{D}{2A}\right)^2}{\frac{CD^2 + AE^2 - 4ACF}{4A^2C}} + \frac{\left(y + \frac{E}{2C}\right)^2}{\frac{CD^2 + AE^2 - 4ACF}{4AC^2}} = 1$$

Since A and C are both positive, $4A^2C$ and $4AC^2$ are both positive. $CD^2 + AE^2 - 4ACF$ must be positive or the conic is degenerate. Thus, we have an ellipse with center $\left(-\frac{D}{2A}, -\frac{E}{2C}\right)$.

(d) $AC < 0 \implies A$ and C have opposite signs. Let's assume that A is positive and C is negative. (If A is negative and C is positive, move the terms to the other side of the equation.) From part (c) we have

$$\frac{\left(x + \frac{D}{2A}\right)^2}{\frac{CD^2 + AE^2 - 4ACF}{4A^2C}} + \frac{\left(y + \frac{E}{2C}\right)^2}{\frac{CD^2 + AE^2 - 4ACF}{4AC^2}} = 1.$$

Since $A > 0$ and $C < 0$, the first denominator is positive if $CD^2 + AE^2 - 4ACF < 0$ and is negative if $CD^2 + AE^2 - 4ACF > 0$, since $4A^2C$ is negative. The second denominator would have the *opposite* sign since $4AC^2 > 0$. Thus, we have a hyperbola with center
$\left(-\frac{D}{2A}, -\frac{E}{2C}\right)$.

8. (a) The first model describes linear motion, whereas the second model describes parabolic motion.

(b) $x = (v_0 \cos \theta)t \implies t = \dfrac{x}{v_0 \cos \theta}$

$y = (v_0 \sin \theta)t \implies t = \dfrac{y}{v_0 \sin \theta}$

$$\frac{x}{v_0 \cos \theta} = \frac{y}{v_0 \sin \theta}$$

$$(v_0 \cos \theta)y = (v_0 \sin \theta)x$$

$$y = (\tan \theta)x$$

$$x = (v_0 \cos \theta)t \implies t = \frac{x}{v_0 \cos \theta}$$

$$y = h + (v_0 \sin \theta)t - 16t^2$$

$$y = h + (v_0 \sin \theta)\left(\frac{x}{v_0 \cos \theta}\right) - 16\left(\frac{x}{v_0 \cos \theta}\right)^2$$

$$y = h + (\tan \theta)x - \left(\frac{16}{{v_0}^2 \cos^2 \theta}\right)x^2$$

(c) In the case $x = (v_0 \cos \theta)t$, $y = (v_0 \sin \theta)t$, the path of the projectile is not affected by changing the velocity v. When the parameter is eliminated, we just have $y = (\tan \theta)x$. The path is only affected by the angle θ.

9. To change the orientation, we can just replace t with $-t$.

$x = \cos(-t) = \cos t$

$y = 2\sin(-t) = -2\sin t$

10. $x = (a - b)\cos t + b\cos\left(\dfrac{a - b}{b}t\right)$

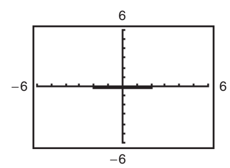
$y = (a - b)\sin t - b\sin\left(\dfrac{a - b}{b}t\right)$

(a) $a = 2, b = 1$

$x = \cos t + \cos t = 2\cos t$

$y = \sin t - \sin t = 0$

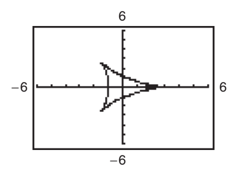
The graph oscillates between -2 and 2 on the x-axis.



(b) $a = 3, b = 1$

$x = 2\cos t + \cos 2t$

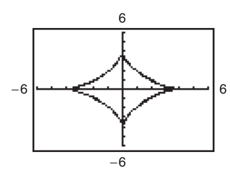
$y = 2\sin t - \sin 2t$



(c) $a = 4, b = 1$

$x = 3\cos t + \cos 3t$

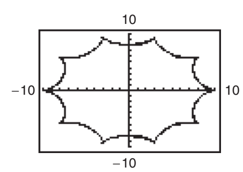
$y = 3\sin t - \sin 3t$



(d) $a = 10, b = 1$

$x = 9\cos t + \cos 9t$

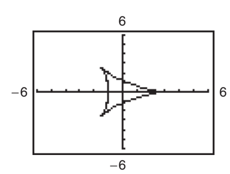
$y = 9\sin t - \sin 9t$



(e) $a = 3, b = 2$

$x = \cos t + 2\cos\dfrac{t}{2}$

$y = \sin t - 2\sin\dfrac{t}{2}$

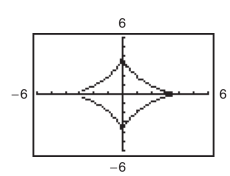


The graph looks the same as the graph in part (b), but is oriented clockwise instead of counterclockwise.

(f) $a = 4, b = 3$

$x = \cos t + 3\cos\dfrac{t}{3}$

$y = \sin t - 3\sin\dfrac{t}{3}$



The graph is the same as the graph in part (c), but is oriented clockwise instead of counterclockwise.

11. (a) $y^2 = \dfrac{t^2(1-t^2)^2}{(1+t^2)^2},\ x^2 = \dfrac{(1-t^2)^2}{(1+t^2)^2}$

$$\frac{1-x}{1+x} = \frac{1-\left(\dfrac{1-t^2}{1+t^2}\right)}{1+\left(\dfrac{1-t^2}{1+t^2}\right)} = \frac{2t^2}{2} = t^2$$

Thus, $y^2 = x^2\left(\dfrac{1-x}{1+x}\right)$.

(b)

$$r^2\sin^2\theta = r^2\cos^2\theta\left(\frac{1-r\cos\theta}{1+r\cos\theta}\right)$$

$$\sin^2\theta(1+r\cos\theta) = \cos^2\theta(1-r\cos\theta)$$

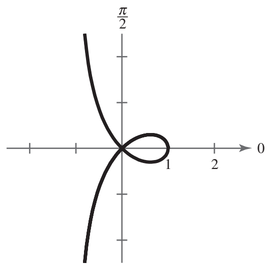
$$r\cos\theta\sin^2\theta + \sin^2\theta = \cos^2\theta - r\cos^3\theta$$

$$r\cos\theta(\sin^2\theta + \cos^2\theta) = \cos^2\theta - \sin^2\theta$$

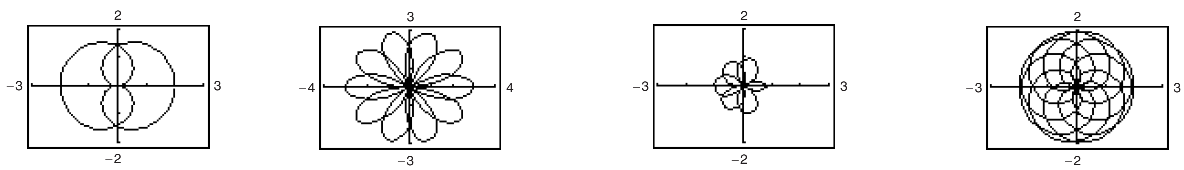
$$r\cos\theta = \cos 2\theta$$

$$r = \cos 2\theta \cdot \sec\theta$$

(c)



12. $r = 2\cos\left(\dfrac{1}{2}\theta\right)$ $\qquad r = 3\sin\left(\dfrac{5\theta}{2}\right)$ $\qquad r = -\cos\left(\sqrt{2}\theta\right)$ $\qquad r = -2\sin\left(\dfrac{4\theta}{7}\right)$



The graphs all contain overlapping loops or petals.

13. $r = a\sin\theta + b\cos\theta$

$r^2 = r(a\sin\theta + b\cos\theta)$

$r^2 = ar\sin\theta + br\cos\theta$

$$x^2 + y^2 = ay + bx$$

$$x^2 + y^2 - bx - ay = 0$$

$$\left(x^2 - bx + \frac{b^2}{4}\right) + \left(y^2 - ay + \frac{a^2}{4}\right) = \frac{a^2}{4} + \frac{b^2}{4}$$

$$\left(x - \frac{b}{2}\right)^2 + \left(y - \frac{a}{2}\right)^2 = \frac{a^2 + b^2}{4}$$

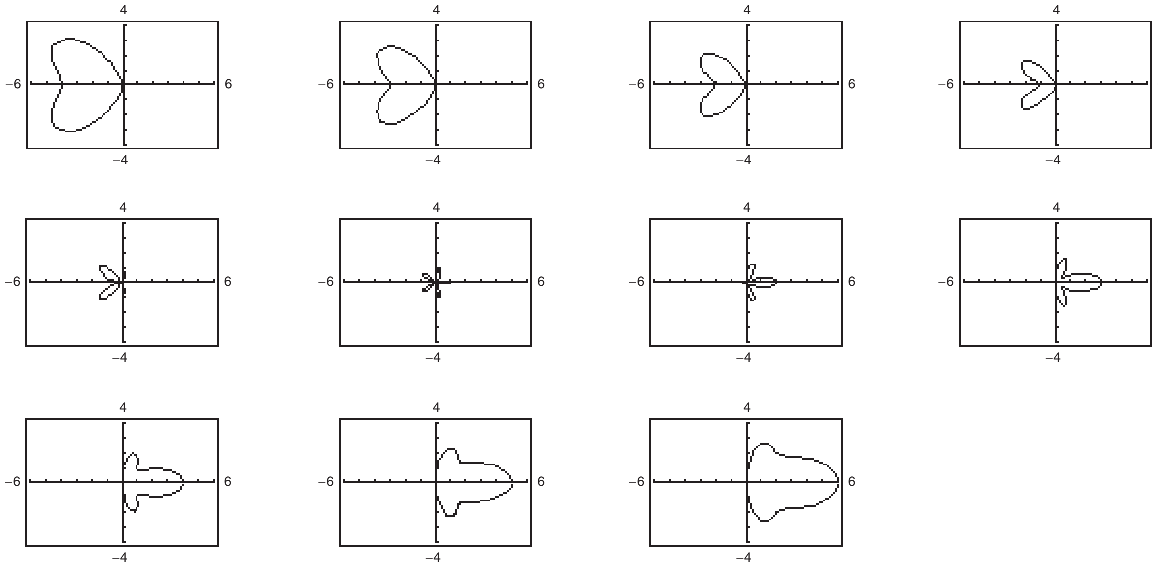
This represents a circle with center $\left(\dfrac{b}{2}, \dfrac{a}{2}\right)$ and radius $r = \dfrac{1}{2}\sqrt{a^2 + b^2}$.

14. $r = e^{\cos\theta} - 2\cos 4\theta + \sin^5\left(\dfrac{\theta}{12}\right)$

(a) No, the graph appears to have a period of 2π but does not. For example, $r(\pi) \neq r(3\pi)$.

(b) By using the table feature of the calculator we have $r \approx 4.077$ when $\theta \approx 5.54$ for $0 \le \theta \le 2\pi$ and $r \approx 4.46$ when $\theta \approx 11.83$ for $0 \le \theta \le 4\pi$. The graph is not periodic. As θ increases the value of r changes.

15.



$n = 1, 2, 3, 4, 5$ produce "bells"; $n = -1, -2, -3, -4, -5$ produce "hearts".

16. (a) Neptune: $a = \dfrac{9.000 \times 10^9}{2} = 4.500 \times 10^9$

$e = 0.0086$

$$r_{\text{Neptune}} = \frac{(1 - 0.0086^2)(4.500 \times 10^9)}{1 - (0.0086)\cos\theta}$$

$$r_{\text{Neptune}} = \frac{4.4997 \times 10^9}{1 - (0.0086)\cos\theta}$$

Pluto: $a = \dfrac{10.0813 \times 10^9}{2} = 5.4065 \times 10^9$

$e = 0.2488$

$$r_{\text{Pluto}} = \frac{(1 - 0.2488^2)(5.4065 \times 10^9)}{1 - (0.2488)\cos\theta}$$

$$r_{\text{Pluto}} = \frac{5.0718 \times 10^9}{1 - (0.2488)\cos\theta}$$

(b) Neptune:

perihelion: $a(1 - e) = 4.500 \times 10^9(1 - 0.0086)$
$= 4.461 \times 10^9$ km

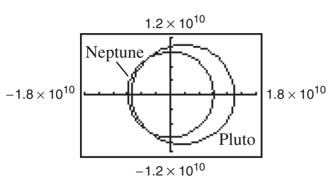
aphelion: $a(1 + e) = 4.500 \times 10^9(1 + 0.0086)$
$= 4.539 \times 10^9$ km

Pluto:

perihelion: $a(1 - e) = 5.4065 \times 10^9(1 - 0.2488)$
$= 4.061 \times 10^9$ km

aphelion: $a(1 + e) = 5.4065 \times 10^9(1 + 0.2488)$
$= 6.752 \times 10^9$ km

(c)



(d) If the orbits were in the same plane, then they would intersect. Furthermore, since the orbital periods differ (Neptune = 164.79 years, Pluto = 247.68 years), then the two planets would ultimately collide if the orbits intersect.

The orbital inclination of Pluto is significantly larger than that of Neptune (17.16° vs. 1.769°), so further analysis is required to determine if the orbits intersect.

(e)

	Neptune	Pluto
perihelion	4.461×10^9 km	4.061×10^9 km

Pluto is sometimes closer to the sun than Neptune (for about 20 years of its 248-year orbit). At the time of its discovery, Pluto was more distant than Neptune. At that time, Pluto was the most distant planet (the ninth in distance) and was also the ninth planet discovered.

Chapter 10 Practice Test

1. Find the angle, θ, between the lines $3x + 4y = 12$ and $4x - 3y = 12$.

2. Find the distance between the point $(5, -9)$ and the line $3x - 7y = 21$.

3. Find the vertex, focus and directrix of the parabola $x^2 - 6x - 4y + 1 = 0$.

4. Find an equation of the parabola with its vertex at $(2, -5)$ and focus at $(2, -6)$.

5. Find the center, foci, vertices, and eccentricity of the ellipse $x^2 + 4y^2 - 2x + 32y + 61 = 0$.

6. Find an equation of the ellipse with vertices $(0, \pm 6)$ and eccentricity $e = \frac{1}{2}$.

7. Find the center, vertices, foci, and asymptotes of the hyperbola $16y^2 - x^2 - 6x - 128y + 231 = 0$.

8. Find an equation of the hyperbola with vertices at $(\pm 3, 2)$ and foci at $(\pm 5, 2)$.

9. Rotate the axes to eliminate the xy-term. Sketch the graph of the resulting equation, showing both sets of axes.

$5x^2 + 2xy + 5y^2 - 10 = 0$

10. Use the discriminant to determine whether the graph of the equation is a parabola, ellipse, or hyperbola.

(a) $6x^2 - 2xy + y^2 = 0$

(b) $x^2 + 4xy + 4y^2 - x - y + 17 = 0$

11. Convert the polar point $\left(\sqrt{2}, \dfrac{3\pi}{4}\right)$ to rectangular coordinates.

12. Convert the rectangular point $\left(\sqrt{3}, -1\right)$ to polar coordinates.

13. Convert the rectangular equation $4x - 3y = 12$ to polar form.

14. Convert the polar equation $r = 5 \cos \theta$ to rectangular form.

15. Sketch the graph of $r = 1 - \cos \theta$.

16. Sketch the graph of $r = 5 \sin 2\theta$.

17. Sketch the graph of $r = \dfrac{3}{6 - \cos \theta}$.

18. Find a polar equation of the parabola with its vertex at $\left(6, \dfrac{\pi}{2}\right)$ and focus at $(0, 0)$.

For Exercises 19 and 20, eliminate the parameter and write the corresponding rectangular equation.

19. $x = 3 - 2 \sin \theta, y = 1 + 5 \cos \theta$

20. $x = e^{2t}, y = e^{4t}$